Apollo Space Program: Saturn V Lunar Trajectory Simulation

Saturn V Stages	Mass m_{fuel}	Mass Frame	t_{final} Burn Duration	Fuel	$v_{exhaust}$ Exhaust Velocity	$R=m_f/t$ Burn Rate	$T=v_{ex}*R$ Thrust
	($x10^6$kg)	($X10^6$kg)	(s)		(m/s)	(kg/s)	($x10^6$N)
First stage	2.04	0.136	150	Kerosene/LOX	2456	13600	33.4
2nd stage	0.428	0.0432	360	LH2/LOX	4220	1190	5.02
3a 1st Burn	0.0356		165	LH2/LOX	4630	216	1
3b TransLunar	0.0674	0.0174	312	LH2/LOX	4630	216	1
Lunar Module	8200kg	2134kg		Aerozine/N2O4	3005	I_{sp}=311s	45kN

First Stage - Lower Atmopshere transport to 38 miles up.

The first Stage consisted of five Rocketdyne F-1 engines, producing 33.4x 10^6 N (7.5 million pounds) of thrust. The duration of the burn is 150 seconds .The fuel was 768,000 liters (203,000 gallons) of RP-1 refined kerosene, and the oxidizer was 1,250,000 liters (331,000 gallons) of liquid oxygen (LOX). The center fifth engine was turned off early to limit the accelration to 4 g's on the astronauts. The reported **final velocity of Stage I was of 2.76 km/s**. The rocket does not maintain a constant vertical ascent pitch angle trajectory, but angles downward into a near horizontal orbital trajectory. The NASA trajectory angle profile had a 10 s delay followed by a linear pitch angle decrease with time. This is simulated See graph below. This affects vertical accleration, avert. At the end of the first Stage the rocket is downrange about 58 miles (93 km) with an altitude of 38 miles. Approximate travel distance of about 69 miles.

Specific Impujlse, Isp, is defined as the number of pound-s of impulse (thrust times duration) given by 1 kg mass of propellant. Kerosene (RP-1) generates an Isp in the range of 270 to 360 seconds, while liquid hydrogen (LH2) engines achieve 370 to 465 seconds.

Stage 1 Velocity Calculation with Atmospheric Drag Data

Saturn V Engine Parameters:

$m_{fuel} := 2.04\cdot 10^6 kg$ $\quad m_{tot} := 2.77\cdot 10^6 kg$ $\quad m_{frame} := 1.36\cdot 10^5 kg$ $\quad t_{burn} := 160s$ $\quad R := 1.289\cdot 10^4\cdot\frac{kg}{s}$

Engine Thrust Change in Momentum, T: $\quad$ T = d(mv)/dt = $v_{exhaust}$ x dm/dt = $v_{exhaust}$ x Fuel Burn Rate

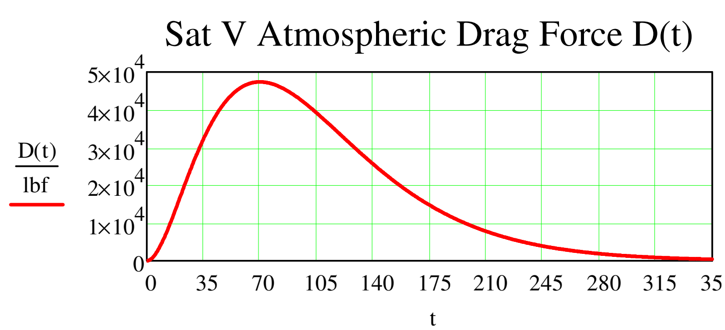
$v_{exhaust} := 2715\frac{m}{s}$ $\quad v_{ex} := v_{exhaust}$ $\quad v_{ex} = 8.907\times 10^3\cdot\frac{ft}{s}$ $\quad T := v_{ex}\cdot R = 3.5\times 10^7 N$

Simulate the pitch angle θ of the rocket's trajectory's during Stage 1 Burn. This angle then determines the resulting g component of the earth's gravitational pull, on the rocket, **in the direction of thrust. Call this g_{thrust}.** Define a parameter, τ, that can be used to adjust the rate and final angle during burn. Adjust τ to match final velocity equal to NASA's Stage 1 velocity. Let the pitch angle, θ, decrease linearly with burn time, from initially 90 degrees vertical to ~ 30 degrees horizontal at the end of Stage 1 burn.
D(t) approximates the Drag. The resultant horizontal acceleration in the direction of travel is then: a(t) - g_{thrust}

$\tau := 400s$ $\quad$ 90° = π/2 radians. $\quad \theta(t) := \frac{\pi}{2}\cdot\left[1 - \Phi(t - 10s)\cdot\left(0.4 + \frac{t}{\tau}\right)\right]$ $\quad g_{thrust}(t) := g\cdot\sin(\theta(t))$

Approximate the rate of fuel consumption,
mass(t), with a linear model: $\quad mass(t) := m_{tot} - R\cdot t$ $\quad D(t) := 3.5\cdot 10^5\cdot lbf\cdot\left(\frac{t}{70}\right)^2\cdot e^{\frac{-t}{35}}$

The Saturn V News Reference, August 1967: "At approximately 69 seconds into the flight the vehicle experiences a condition of maximum dynamic pressure. At this time, the restraining drag force is approximately equal to 460,000 pounds." The atmosphere ends at about 100km or about 200s into the flight. Thus, we approximate the **Saturn V Atmospheric Drag Force, D(t),** as a function of recorded time as shown at right.



Stage 1 Simulation Equations for Acceleration, Velocity, and Distance:

NASA's velocity was 2760 m/s

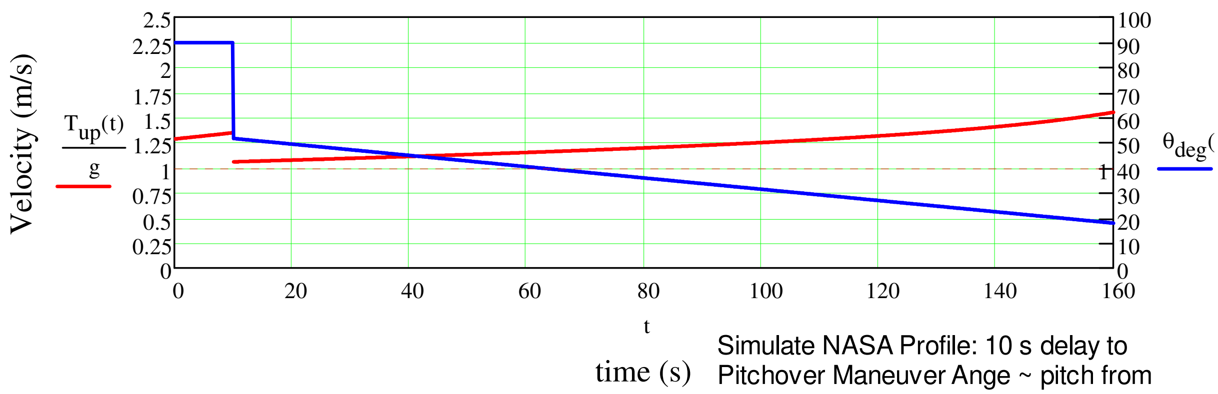
$$a(t) := \frac{T - D\left(t \cdot s^{-1}\right)}{mass(t)} \qquad v1_{final} := \int_0^{t_{burn}} \left(a(t) - g_{thrust}(t)\right) dt \qquad v1_{final} = 2.763 \times 10^{3} \frac{m}{s}$$

$$a(0) = 1.288 \cdot g \qquad a(11s) = 1.357 \cdot g \qquad a\left(t_{burn}\right) = 5.031 \cdot g \qquad v1_{final} = 6.181 \times 10^{3} \cdot mph$$

Check the Simulation to make sure that the vertical thrust, Tup, is always greater than 1 g as Pitch θ is reduced.

$$T_{up}(t) := a(t) \cdot \sin(\theta(t)) \qquad T_{up}(10.1s) = 1.061 \cdot g$$

Stage 1 Upward Thrust (red) and Pitch Profile (blue) vs. time



Simulate NASA Profile: 10 s delay to Pitchover Maneuver Ange ~ pitch from 45 to 20 degrees angle to horizontal. Note: This comes from "eyeballing" NASA's flight data.

NASA's Total Distance from lauch point by Stage 1: $travel := \sqrt{(58mile)^2 + (38mile)^2} = 69.34 \cdot mile$

$38mile = 61.155 \cdot km$

The calculated thrust distance is 89 miles, approximately equal to NASA's Stage 1 Travel Distance of 69 miles.

$$distance := \int_0^{t_{burn}} \int_0^{t} a(t) - g_{thrust}(t)\, dt\, dt \qquad distance = 89.397 \cdot mile$$

NASA Stage 1 height is 61 km.

Calculate the Stage 1 Height

$$height := \int_0^{t_{burn}} v1_{thrust}(t) \cdot \sin(\theta(t))\, dt$$

$$height = 65.786\,km$$

Calculate the Stage 1 Range $\qquad R_e := 6371km$

$$range := \int_0^{t_{burn}} v1_{thrust}(t) \cdot \cos(\theta(t)) \cdot \frac{R_e}{\left(R_e + 191km\right)}\, dt$$

$$range = 122.799\,km$$

Calculate the Stage 1 Specific Impulse, I_{SP}:

Total Impule, I $\quad I := T \cdot t_{burn} = 5.599 \times 10^{9} \frac{m \cdot kg}{s}$ $\qquad$ Specific Impulse, Isp $\quad I_{sp} := \frac{I}{m_{fuel} \cdot g} = 279.893\,s$

Tsiokolvsky's Equation for Stage 1 Δv: $\quad \Delta v := v_{exhaust} \cdot \ln\left(\frac{m_{tot}}{m_{tot} - m_{fuel}}\right) \qquad \Delta v = 3.621 \times 10^{3} \frac{m}{s}$

Stage 2 Burn: Velocity Calculation - Assuming 20 Degree Ascent Angle & ~ 0.35g

NASA's velocity at the end of Stage 2 was **6995 km/s** at 191 km alititude. At this altitude, drag is not significant.

$m_{fuel} := 4.28 \cdot 10^5 kg \qquad m_{tot} := 5.94 \cdot 10^5 kg \qquad v_{ex2} := 4220 \frac{m}{s} \qquad R := 1190 \cdot \frac{kg}{s}$

$\theta_{deg}(150) = 20.25 \qquad g \cdot \sin\left(\frac{32}{90} \cdot \frac{\pi}{2}\right) = 0.53 \cdot g$

$g_{thrust}(199s) = 0.16 \cdot g \qquad \left(1 - \frac{199s}{\tau}\right) \cdot 90 = 45.225 \qquad v_{ex2} = 1.385 \times 10^4 \cdot \frac{ft}{s} \qquad T2 := v_{ex2} \cdot R = 5.022 \times 10^6 N$

$mass(t) := m_{tot} - R \cdot t \qquad a2(t) := \frac{T2}{mass(t)} - g_{thrust}(150s) \qquad t_{burn} := 360s \qquad v2_{final} := v1_{final} + \int_0^{t_{burn}} a2(t)\, dt$

$g_{thrust}(150s) = 0.346 \cdot g \qquad a2(0s) = 5.06 \frac{m}{s^2} \qquad a2(t_{burn}) = 26.931 \frac{m}{s^2} \qquad v2_{final} = 6.932 \times 10^3 \frac{m}{s}$

Stage 3 Burn: Final Velocity Calculation - Assume 22 Degree Ascent & ~ 0.47g

Stage 3 is close to the altitiude and orbital velocity,
where gravity, rather than pulling the rocket down,
is providing the centripetal acceleration needed to keep
the rocket in a curved or orbital path around the earth.

At an altitude of 191.2 km, Apollo 11 went into a parking orbit.
The stated NASA stage 3 velocity was 7.791 km/s. C

Calculate the Required Velocity to go into Orbit at an Altitude of 191 km

$G := 6.67 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2} \qquad M_e := 6 \cdot 10^{24} kg \qquad R_e := 6371km \qquad v_{orbital} := \sqrt{G \cdot \frac{M_e}{R_e + 191km}} = 7.809 \times 10^3 \frac{m}{s}$

Stage 3 Vehicle Parameters

$m_{fuel} := 1 \cdot 10^5 kg \qquad m_{tot} := 1.23 \cdot 10^5 kg \qquad v_{ex3} := 4630 \frac{m}{s} \qquad R := 216 \cdot \frac{kg}{s} \qquad t_{burn} := 150s$

$v_{ex3} = 1.519 \times 10^4 \cdot \frac{ft}{s} \qquad T := v_{ex3} \cdot R = 1 \times 10^6 N$

$mass(t) := m_{tot} - R \cdot t \qquad a3(t) := \frac{T}{mass(t)} - g_{thrust}(160s) \qquad v3_{final} := v2_{final} + \int_0^{t_{burn}} a3(t)\, dt$

$a3(0s) = 5.1 \frac{m}{s^2} \qquad a3(t_{burn}) = 8.008 \frac{m}{s^2} \qquad v3_{final} = 7.893 \times 10^3 \frac{m}{s}$

$mass(t_{burn}) = 9.06 \times 10^4 kg \qquad v3_{final} = 1.766 \times 10^4 \cdot mph$

Earth's Escape Velocity from Surface of Earth:

$$v_{escape} := \sqrt{\frac{2G \cdot M_e}{R_e}} = 1.121 \times 10^4 \frac{m}{s}$$

Atmospheric Parameters

Atmosphere model: US 1976 Standard Atmosphere (z < 1000 m):

Temperature (K) -> $T(z) := 288.149 - .00649 \cdot z$

Pressure (Pa) -> $p(z) := 101325 \cdot \left(1 - 2.26 \cdot 10^{-5} \cdot z\right)^{5.25}$

Density (kg/m³) -> $\rho(z) := \dfrac{p(z)}{T(z) \cdot 287}$

Speed of Sound (m/sec) -> $v_s(z) := \sqrt{401.8 \cdot T(z)}$

Rocket geometry:

Body diameter (m) -> $d := 10$

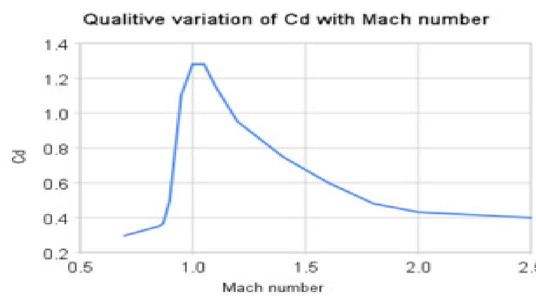
Reference area (m²) -> $S := \pi \cdot \left(\dfrac{d}{2}\right)^2$ $\quad S = 78.54$

Drag model:

Inport drag data from MISSILE DATCOM -> $\text{Mach} := \begin{pmatrix} .2 \\ .4 \\ .6 \\ .8 \end{pmatrix}$ $\quad C_D := \begin{pmatrix} .41 \\ .394 \\ .379 \\ .379 \end{pmatrix}$

Interpolate for drag vs Mach # -> $C_d(M) := \text{interp}\left(\text{cspline}\left(\text{Mach}, C_D\right), \text{Mach}, C_D, M\right)$

$M := 0, .001 .. 10$



Drag Equation

C_d determines complex dependencies

$D = C_d \, \rho \, v^2 A/2$