I. Simple Lunar Trajectories: Kepler's Elliptical Model (Planar Point Mass)

This Section on Kepler is shown for historical interest. Newton's Dynamics is used in all the following Sections

Kepler's E Model (Planar Point Mass 2 Body): **See** the **Glossary** and **Figures** in **last two pages** of this Study

Convert Cartesian Ellipse Eq. in (x,y) to polar (r,v) coordinates
Ellipse is relative to the **focus**

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad r(\nu, e) := \frac{a\cdot\left(1-e^2\right)}{1+e\cdot\cos\left(\nu-\frac{\pi}{2}\right)}$$

$x(a,\theta) := a\cdot\cos(\theta)$ and $y(b,\theta) := b\cdot\sin(\theta)$

$0 \le t < 2\pi$ $\qquad e = \frac{c}{a}$ $\qquad r(x,y) := \sqrt{x^2+y^2}$ and $\theta(x,y) := \mathrm{atan}\left(\frac{y}{x}\right)$

For the moon

$e_m := .0549$ $\quad d_m := 384400\mathrm{km}$ $\quad d_{ap} := 406603\mathrm{km}$ $\quad m_m := 7.347\cdot 10^{22}\mathrm{kg}$ $\quad a_m := \frac{d_{ap}}{1+e}$ $\quad \mu := 3.986\cdot 10^5 \frac{\mathrm{km}^3}{\mathrm{sec}^2}$

For the Earth: Mass $\quad m_e := 5.972\cdot 10^{24}\mathrm{kg}$ $\qquad$ Note: a and b are distances from the center, c

The parameter e is known as the eccentricity. The value of this parameter defines the shape of our orbit. Depending on the value of e there are four kinds of shapes (conic sections), which means there are four kinds of orbits: circle, ellipse, parabola, and hyperbola, for e = 0, < 1, = 1, and > 1, respectively.

$H \equiv 1$ $\quad e_e := .6$ $\quad e_c := 0$ $\quad e_h := 2$ $\quad e_p := 1.000$ $\quad \theta := 0, 0.01 .. 2\pi$ $\qquad G\cdot m_e = 3.985\times 10^{14}\frac{\mathrm{m}^3}{\mathrm{s}^2}$

Basics from Newton's Laws: Energy, Momentum, Parameters of Ellipse $\qquad r_o := 300\mathrm{km}$ $\quad \phi_o := 0$

$\mathrm{Energy}(v,r) := \frac{v^2}{2} - \frac{\mu}{r}$ $\qquad h\left(v_o, r_o, \phi_o\right) := r_o\cdot v_o\cdot\cos\left(\phi_o\right)$ $\qquad h = r^2\cdot v\nu^2$ $\qquad h_u(p) := \sqrt{\mu\cdot p}$

$p\left(v_o\right) := \frac{h\left(v_o, r_o, \phi_o\right)^2}{\mu}$ $\qquad a\left(v_o\right) := \frac{-\mu}{\mathrm{Energy}\left(v_o, r_o\right)}$ $\qquad e_{traj}(v,a) := \sqrt{1-\frac{p(v)}{a}}$

Period of Moon Sat Orbit

$T := 2\cdot\pi\cdot\sqrt{\frac{a_m{}^3}{G\cdot m_e}} = 99.98\cdot\mathrm{hr}$

$$r_h(\theta, e) := \frac{H}{1+e\cdot\cos\left(\theta+\frac{\pi}{2}\right)}$$

If we can Solve for Eccentric Anomaly, E, we get Time of Flight, TOF, t - T

$\cos(\nu) = \frac{p-r}{e\cdot r}$ $\qquad \nu(p,r,e) := \mathrm{acos}\left(\frac{p-r}{e\cdot r}\right)$

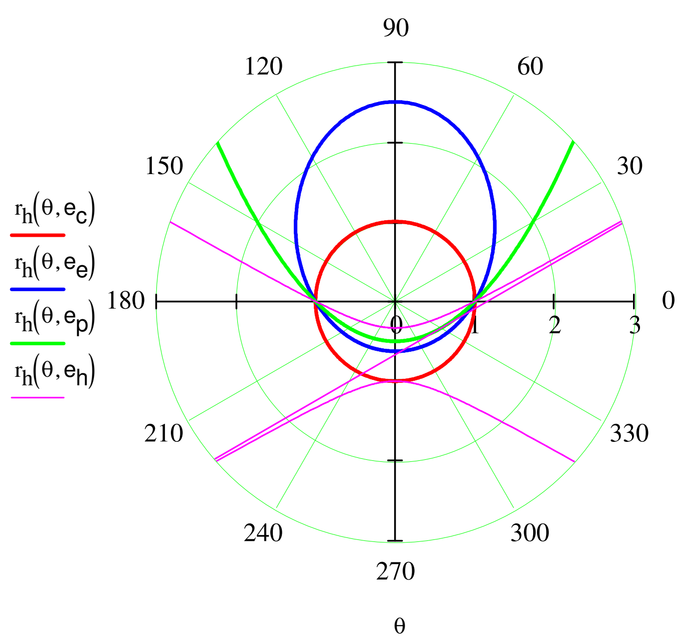
Recursion for Eccentric Anomaly, M & E (Deg)

mean anomaly M (in deg (0<= M<=360)

$MA\left(M_o, t, t_o\right) := M_o + \sqrt{\frac{\mu}{a_m{}^3}}\cdot\left(t-t_o\right)$

```
EcA(e,M,dp) := | mx_it ← 30
               | i ← 0
               | K ← π/180
               | del ← 10^(−dp)
               | m ← M/360
               | m ← 2·π(m − floor(m))
               | E ← m  if  e < 0.8
               | E ← π  otherwise
               | F ← E − e·sin(m) − m
               | while |F| > del ∧ i < mx_it
               |     | E ← E − F/(1 − e·cos(E))
               |     | F ← E − e·sin(E) − m
               |     | i ← i + 1
               | E ← E/K
```

Plot of Conic Orbits: c, e, p, h



Find E and φ In Degrees

t/T: $\frac{27}{360} = 0.075$ $\qquad \mathrm{EcA}\left(e_m, 27, 5\right) = 28.501$ $\qquad \phi(e, \mathrm{EcA}) := 90 - \frac{180\cdot\mathrm{atan2}\left(\sqrt{1-e^2}\cdot\sin(\mathrm{EcA}), \cos(\mathrm{EcA})-e\right)}{\pi}$

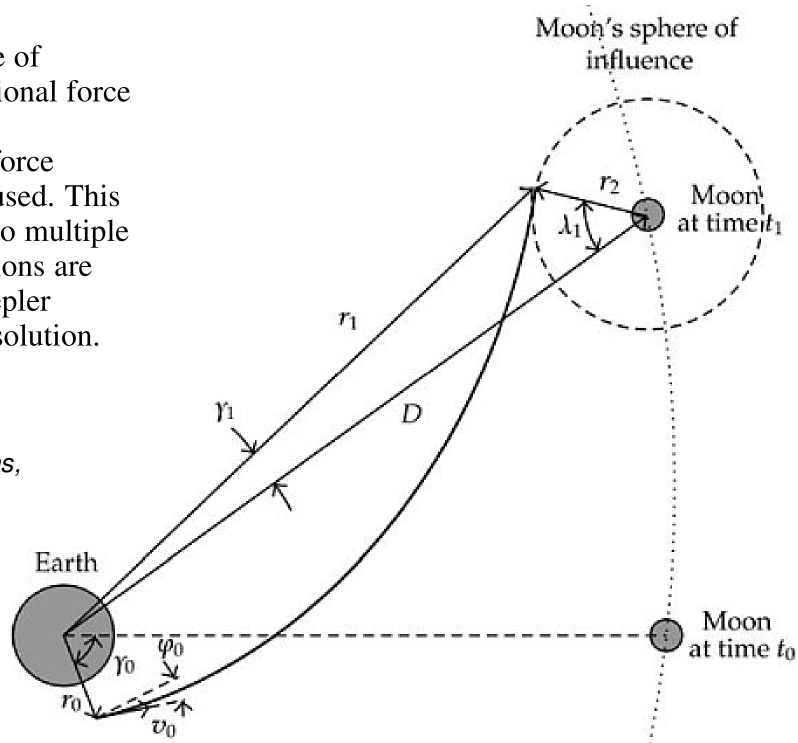
$\phi(0.977, 48.43418\cdot\mathrm{deg}) = 153.029$

The Patched Conic Section Approximation for Finding a Lunar Trajectory

The Patched Conic Method is an Approximation for finding a trajectory by dividing space between the sphere of influence (SOI) of the earth, Lunar Earth Orbit (LEO) and the SOI region of the moon.

When the spacecraft is within the sphere of influence of the moon, only the gravitational force between the spacecraft and the moon is considered, otherwise the gravitational force between the spacecraft and the earth is used. This reduces a complicated n-body problem to multiple two-body problems, for which the solutions are the well-known conic sections of the Kepler orbits. Below is an example composite solution.

See for Example:
Optimal Two-Impulse Trajectories with Moderate Flight Time for Earth-Moon Missions,
Sandro da Silva Fernandes
Mathematical Problems in Engineering
Vol. 2012, Article ID 971983,
or
Bate, R. R., D. D. Mueller, and J. E. White,
Fundamentals of Astrodynamics



Rather than dealing with large powers of 10, we can use **Astronomical Units**, for distance, velocity, time: AU, VU, TU. Where AU is the mean distance of the earth to the sun and DU is the radius of the earth. TU is the time unit. Then the velocity unit, (VU) is equal to DU/TU.

$DU := 6378.145km$ $\quad AU := 1.496\cdot 10^{8}km$ $\quad kmps := \frac{km}{s}$ $\quad VU := 7.905368kmps$ $\quad TU := 806.8s$ $\quad D := d_m$

Laplace's Equation for Moon's Sphere of Influence:
this is about 1/6 of the distance, D, to the moon

$R_{sif} := D\cdot\left(\frac{m_m}{m_e}\right)^{0.4}$ $\quad R_s := 66300km$ $\quad R_s = 10.395\cdot DU$

The conic patched problem for finding a trajectory can be stated as follows:

Given: Initial rocket launch conditions in the earth's sphere of Influence, that is, initial position, velocity, flight path angle, and phase angle: $\mathbf{r_0, v_0\ \phi_0}$, **and** $\boldsymbol{\gamma_0}$,
The three quntities $\mathbf{r_0, v_0\ \phi_0}$ will give us initial energy and anglular momentum.

Find: Arrival conditions at moon's Sphere of Influence: $\mathbf{r_1, v_1\ \phi_1, \lambda_1}$.

$\mathbf{r_0, v_0\ \phi_0}$, **and** $\boldsymbol{\lambda_1}$

The problem with assigning these initial points is that they may not give a satisfactory solution to match the arrival conditions. Our strategy is to use the arrival ange λ_1 to the moon's SOI as one of the independent condition .

Given the 3 initial conditions and one arrival condition as our **independent variables:**
These will move us into the radius of the moon's sphere of influence. Some trial and error may still be required.

EXAMPLE: See Bate, R. R., D. D. Mueller, and J. E. White, *Fundamentals of Astrodynamics*

Solution: Select the **Apollo 11 Flight Conditions** for initial conditions: $\mathbf{r_0}$, $\mathbf{v_0}$, $\mathbf{\phi_0}$ **and** $\mathbf{\lambda_1}$.

Given: $r_0 := DU + 334km$ $v_0 := 10.6kmps$ $\phi_0 := 0deg$ A reasonable angle to arrive at moon $\lambda_1 := 30deg$

Find: $\mathbf{r_1}$, $\mathbf{v_1}$, $\mathbf{\phi_1}$, $\mathbf{\gamma_1}$ (the last symbol, γ, is the Greek letter gamma, the Arrival Phase Angle at the Moon)

Initial Energy and Angular Momentum are $Energy(v_0, r_0) = -0.011 \cdot VU^2$ $h_0 := h(v_0, r_0, \phi_0) = 1.441 \cdot \frac{DU^2}{TU}$

$D = 60.268 \cdot DU$ By the Law of Cosines: $r_1(\lambda_1) := \sqrt{D^2 + R_s^2 - 2D \cdot R_s \cdot \cos(\lambda_1)}$ $r1 := r_1(\lambda_1) = 51.529 \cdot DU$

From Law of Conservation of Energy and Momentum: $E_0 := Energy(v_0, r_0)$ $E_0 = -0.011 \cdot \frac{DU^2}{TU^2}$ $h_1 := h_0$

$v_1(r_1) := \sqrt{2 \cdot \left(E_0 + \frac{\mu}{r_1}\right)}$ $v1 := v_1(r1) = 0.128 \cdot VU$ $v1 := 0.1296VU$ $\phi_1 := acos\left(\frac{h_1}{r1 \cdot v1}\right)$ $\phi_1 = 77.542 \cdot deg$

In order to calculate the **Time of Flight**, TOF, to the moon's SOI, we need to Find:

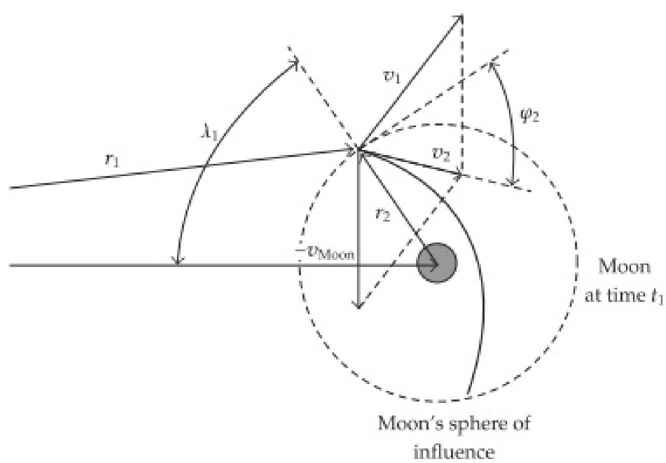
p, a, e, $\mathbf{E_0}$ **and** $\mathbf{E_1}$ for the Geocentric Trajectory.

$p := \frac{h_0^2}{\mu} = 2.075 \cdot DU$ $a := \frac{-\mu}{2\,Energy(v_0, r_0)}$ $e := \sqrt{1 - \frac{p}{a}}$ $e = 0.977$ $\nu_1 := \nu(p, r1, e)$ $\nu_1 = 2.956$

$\gamma_1 := asin\left(\frac{R_s}{r1} \sin(\lambda_1)\right) = 5.789 \cdot deg$ $a = 44.698 \cdot DU$ since: $\nu_0 := 0$ $EcA_0 := 0$ $EcA_1 := acos\left(\frac{e + \cos(\nu_1)}{1 + e \cdot \cos(\nu_1)}\right)$

$EcA_1 = 1.728$ $TOF := \sqrt{\frac{a^3}{\mu}} \cdot \left[\left(EcA_1 - e \cdot \sin(EcA_1)\right) - \left(EcA_0 - e \cdot \sin(EcA_0)\right)\right]$ $TOF = 51.132 \cdot hr$

We can use the same procedure at the moon (Selenocentric).
See Section XVI for the Newtonian Gravitational Solution for the Lunar Trajectory.
We need to determine the values of v1 and Rs in units based on the moon's gravitational attraction parameters.
The Angular Velocity of the Moon (ω_m) in its orbit is



: $\omega_m := 2.649 \cdot 10^{-6} \frac{rad}{s}$ $\omega_m := 2.137 \cdot 10^{-3} \frac{1}{TU}$ $\gamma_0 := \nu_1 - \nu_0 - \gamma_1 - \omega_m \cdot TOF$ $\gamma_0 = 135.637 \cdot deg$

$v_{1m} := 1.024kmps$ $\mu_m := 4093 \frac{km^3}{s^2}$ $v_m := 1.018kmps$ Then $v_{2m} := 1.198kmps$

$\varepsilon_2 := 5.68deg$ $e_m := 2.078$ $r_p := 4105km$ $h_p := 2367km$ $R_s = 10.395 \cdot DU$

$\mu_m := 4.903 \cdot 10^3 \cdot \frac{km^3}{s^2}$

Time of Flight

Develop an algorithm to Calculate Time of Flight

$$TOF_{alg}(v_0, r_0, \phi_0, \lambda_1) := \left| \begin{array}{l} h_0 \leftarrow r_0 \cdot v_0 \cdot \cos(\phi_0) \\ p \leftarrow \dfrac{{h_0}^2}{\mu} \\ E_0 \leftarrow \mathrm{Energy}(v_0, r_0) \\ EcA_0 \leftarrow 0 \\ a \leftarrow \dfrac{-\mu}{2E_0} \\ e \leftarrow \sqrt{1 - \dfrac{p}{a}} \\ r1 \leftarrow \sqrt{D^2 + {R_s}^2 - 2D \cdot R_s \cdot \cos(\lambda_1)} \\ \nu_1 \leftarrow \nu(p, r1, e) \\ EcA_1 \leftarrow \mathrm{acos}\left(\dfrac{e + \cos(\nu_1)}{1 + e \cdot \cos(\nu_1)}\right) \\ TOF \leftarrow \dfrac{\sqrt{\dfrac{a^3}{\mu}} \cdot \left[\left(EcA_1 - e \cdot \sin(EcA_1)\right) - \left(EcA_0 - e \cdot \sin(EcA_0)\right)\right]}{hr} \\ A \leftarrow (TOF \quad e)^T \end{array} \right.$$

This gives a different value

$v_0 = 10.846 \cdot kmps$ $\quad TOF_{alg}(v_0, r_0, \phi_0, \lambda_1) = \begin{pmatrix} 51.132 \\ 0.977 \end{pmatrix}$

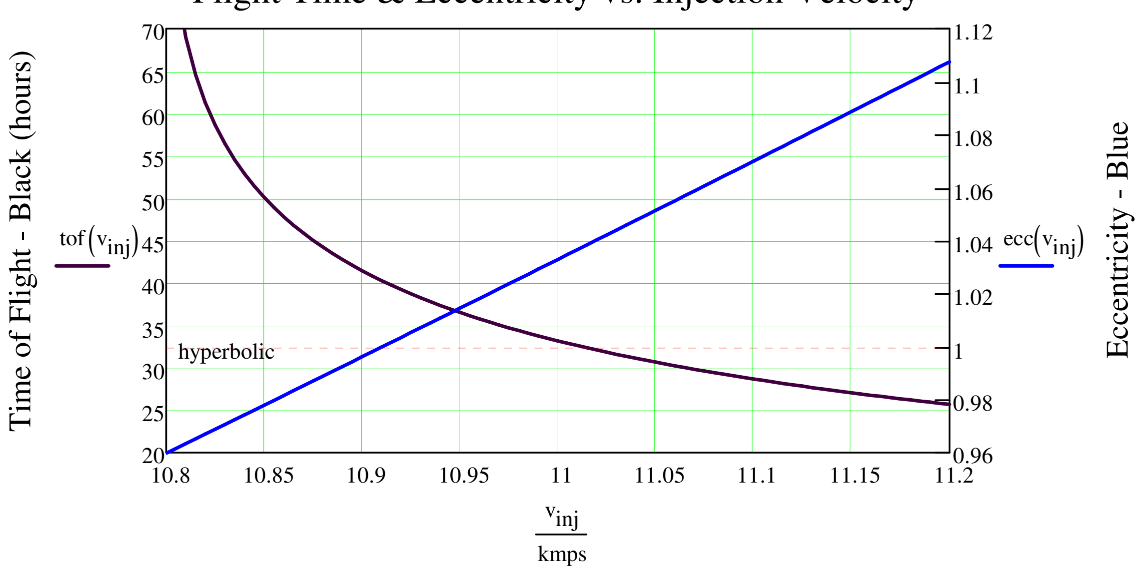
$tof(v_0) := TOF_{alg}(v_0, r_0, \phi_0, \lambda_1)_0$ $\quad ecc(v_0) := TOF_{alg}(v_0, r_0, \phi_0, \lambda_1)_1$

Initial Conditions: $r_0 = 1.05 \cdot DU$ $\quad Alititude := r_0 - 1DU = 318.907 \cdot km$ $\quad \phi_0 = 0$

$hyperbolic := 1$ $\quad v_{inj} := 10.8kmps, 10.805kmps .. 11.2kmps$

Note: As the velocity increases above the minimum 10.8 kmps, the Time of Flight decreases and the trajectory shape changes from Elliptical to Hyperbolic.

Flight Time & Eccentricity vs. Injection Velocity



kmps

Polar Plot of the Solution for the Patched Conic Lunar Approximation

$\nu := -90.002deg, -90.001deg .. 41deg$ $\qquad \chi := 39.5deg, 39.501deg .. 360deg$ $\qquad r(\nu) := \frac{a \cdot (1 - e^2)}{1 + e \cdot \cos(\nu + \gamma_0)}$

Note: λ_1 is not = 30 deg $\qquad \varphi := 33deg$ $\qquad Earth(\theta) := 1.5 \sin(\theta + \varphi)$ $\qquad \theta := 0, 0.001 .. 2\pi$

$r_m := 82 \qquad a_m := 1.5$ $\qquad r_{moon}(\theta, \varphi) := r_m \cdot \cos(\theta - \varphi) + \sqrt{a_m{}^2 - r_m{}^2 \sin(\theta - \varphi)^2}$

Radius of Moon Sphere of Influence $\qquad r_{msi}(\theta, \varphi) := r_m \cdot \cos(\theta - \varphi) + \sqrt{10.4^2 - r_m{}^2 \sin(\theta - \varphi)^2}$

$\xi := 0.05, 0.051 .. \varphi - 0.05$ $\qquad r_{m\_path}(\xi) := r_m$

Point of Conic Patch

$SpShip := 75.5$

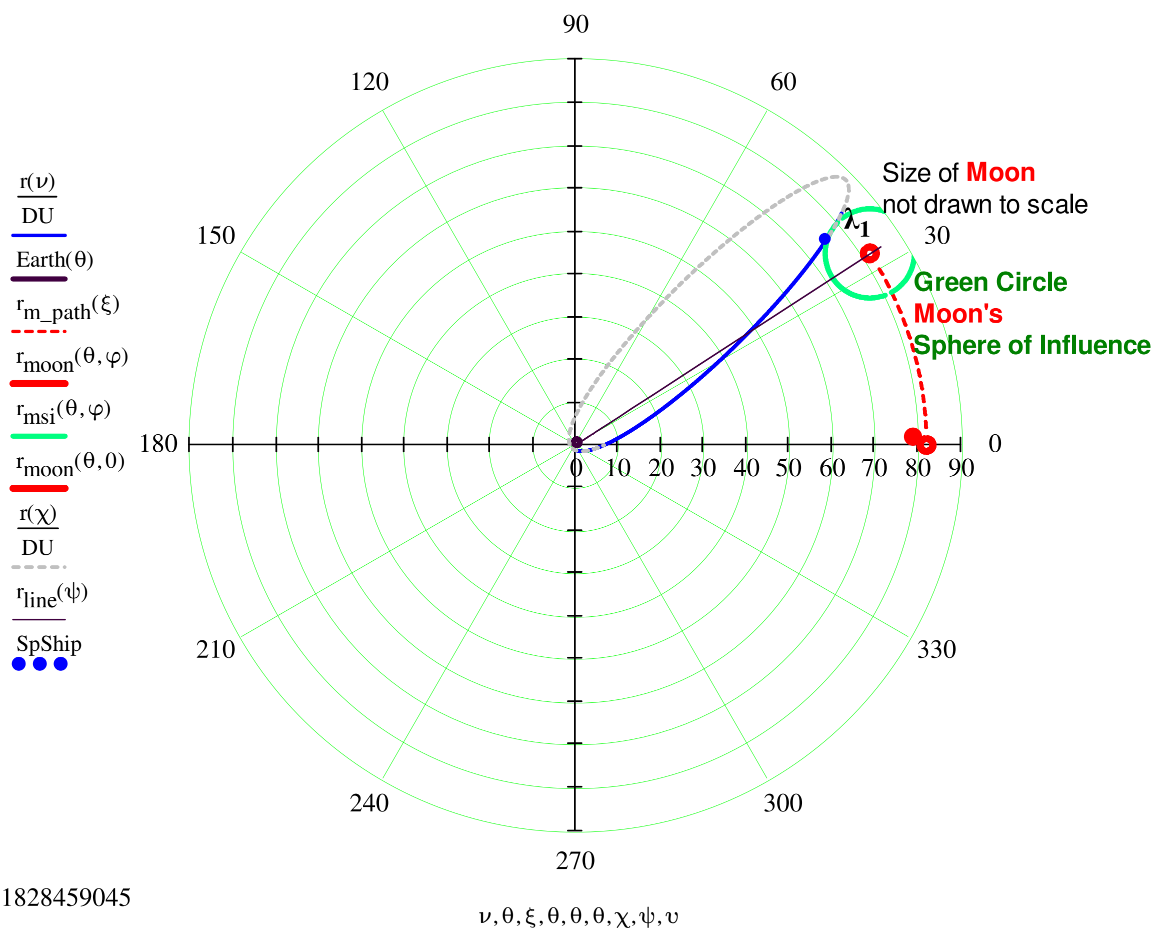
$\psi := 0, 0.0017365 .. \varphi$ $\qquad r_{line}(\theta) := \frac{0.1}{\sqrt{1 - (1 \cdot \cos(\theta - \varphi))^2}}$ $\qquad \upsilon := 39.5deg$

Polar Plot: Geocentric Frame - Earth at the Center

From the list of functions shown on the left of the plot below:

r(v) shows the Trajectory Ellipse Conic Ptach in blue, Earth(θ) is at the center in black, $r_{moon}(\theta,\varphi)$ in red is the location of the moon at intercept $\varphi = 33^o$, $r_{msi}(\theta)$ is the circle in green of the moon's of sphere of influence, $r_{moon}(\theta,0)$ in red is the initial location of the moon at 0^o, $r_{m\_path}(\xi)$ is the dotted line path of moon from 0 to φ. r(χ) is the dotted line that shows the elliptical path back to the earth, and r_{line} is the red straight line from earth at center to the moon to show angle λ_1. SpCraft is where SpaceCraft enters the Moon's Sphere of Influence. Point of Conic Patch. Blue dot.

Patched Conic Approx. Trajectory to Moon (Red)



$e := 2.718281828459045$

IA. Apollo Free Return Trajectory: Simulation for CSM to Moon & Back

Trajectory Model: 3-Body (Earth, Moon, Spacecraft) 2D Planar Point Mass with **Earth at Center**

This 3 body gravitational solution for the FRT uses the Mathcad Differential Equation Solving Methodology discussed: arXiv:1504.07964
"***Motion of the planets: the calculation and visualization in Mathcad***", Valery Ochkov, Katarina Pisa

The aborted Apollo 13 mission was the only mission to actually turn around the Moon in a free-return trajectory.

Solve the Gravitational and Dynamics Equations for Earth, Moon, & CSM Trajecto

$kg := 1 \quad m := 1 \quad s := 1 \quad N := 1 \quad s := 1 \quad min := 60s \quad hr := 3600s \quad kgf := 9.80665N$

$km := 1000m \quad kmps := km \quad kph := \frac{km}{hr} \quad mph := 0.447 \cdot 10^{-3} kmps \quad G := 6.67384 \cdot 10^{-11} \cdot \frac{N \cdot m^2}{kg}$

Run Simulation for 160 hrs **Apollo 11 Orbit 77 hrs**

$FRAME := 999 \quad n_{ode} := 20000 \quad n := 999 \quad n_{plot} := 10000 \quad t_{end} := \frac{160hr}{n+1} \cdot (FRAME + 1) \quad t_{orb} = 81.44\ hr$

Time of Flight (TOF) = t_{orb}

Trajectory to Moon's Sphere of Influence

Initial x,y Velocity CSM **Radius of Earth** **Apogee to Moon**

$v_{0x} := 6.811kmps \quad v_{0y} := 6.356kmps \quad v_{CSM} := 9.317kmps \quad R_e := 6370km \quad d_{m\_ap} := 405500km$

Define Gravitational and Dynamics Equations for Earth, Moon, and CSM

$$\begin{array}{l} \\ \text{Earth, e} \\ \text{Moon, m} \\ \text{CSM, s} \end{array} \begin{array}{c} \text{Mass Start position Start Velocity} \\ \begin{pmatrix} m_e & x_{e0} & y_{e0} & vx_{e0} & vy_{e0} \\ m_m & x_{m0} & y_{m0} & vx_{m0} & vy_{m0} \\ m_s & x_{s0} & y_{s0} & vx_{s0} & vy_{s0} \end{pmatrix} \end{array} := \begin{pmatrix} 5.972 \cdot 10^{24} kg & 0m & 0m & 0kph & 0kph \\ 7.347 \cdot 10^{22} kg & d_{m\_ap} & 0km & 0kmps & 0.97kmps \\ 13600kg & R_e + 100km & R_e - 100km & v_{0x} & v_{0y} \end{pmatrix}$$

Given **Solve Set of Differential Guidance Equations for 3 Body Problem of Earth, Moon, and CSM**

$x_e(0) = x_{e0} \quad x_e'(0) = vx_{e0} \quad y_e(0) = y_{e0} \quad y_e'(0) = vy_{e0}$

$$m_e \cdot x_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (x_m(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (x_s(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3}$$

$$m_e \cdot y_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (y_m(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (y_s(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3}$$

$x_m(0) = x_{m0} \quad x_m'(0) = vx_{m0} \quad y_m(0) = y_{m0} \quad y_m'(0) = vy_{m0}$

$$m_m \cdot x_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (x_e(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_s \cdot (x_s(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3}$$

$$m_m \cdot y_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (y_e(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G\, m_m \cdot m_s \cdot (y_s(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3}$$

$x_s(0) = x_{s0} \quad x_s'(0) = vx_{s0} \quad y_s(0) = y_{s0} \quad y_s'(0) = vy_{s0}$

$$m_s \cdot x_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (x_e(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (x_m(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3}$$

$$m_s \cdot y_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (y_e(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (y_m(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3}$$

IA. Free Return Trajectory: 3 Body Sim for CSM to the Moon & Back

Trajectory Model: 3-Body (Earth, Moon, Spacecraft) 2D Planar Point Mass with **Earth at Center**

Differential Equation Solver

$$\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \end{pmatrix}, t, t_{end}, n_{ode}\right]$$

Initial Velocity (km/s) of CSM at an Altitude of 141 km:

$$\sqrt{v_{0x}^2 + v_{0y}^2} = 9.316\cdot\text{kmps}$$

Distance to the Center of the Moon

$$d_m(t) := \left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right] \qquad \frac{d_m(t_{orb})}{R_m} = 1.619$$

time t_{fly}, just beyond lunar fly by time $\quad$ **t_{flydot}**

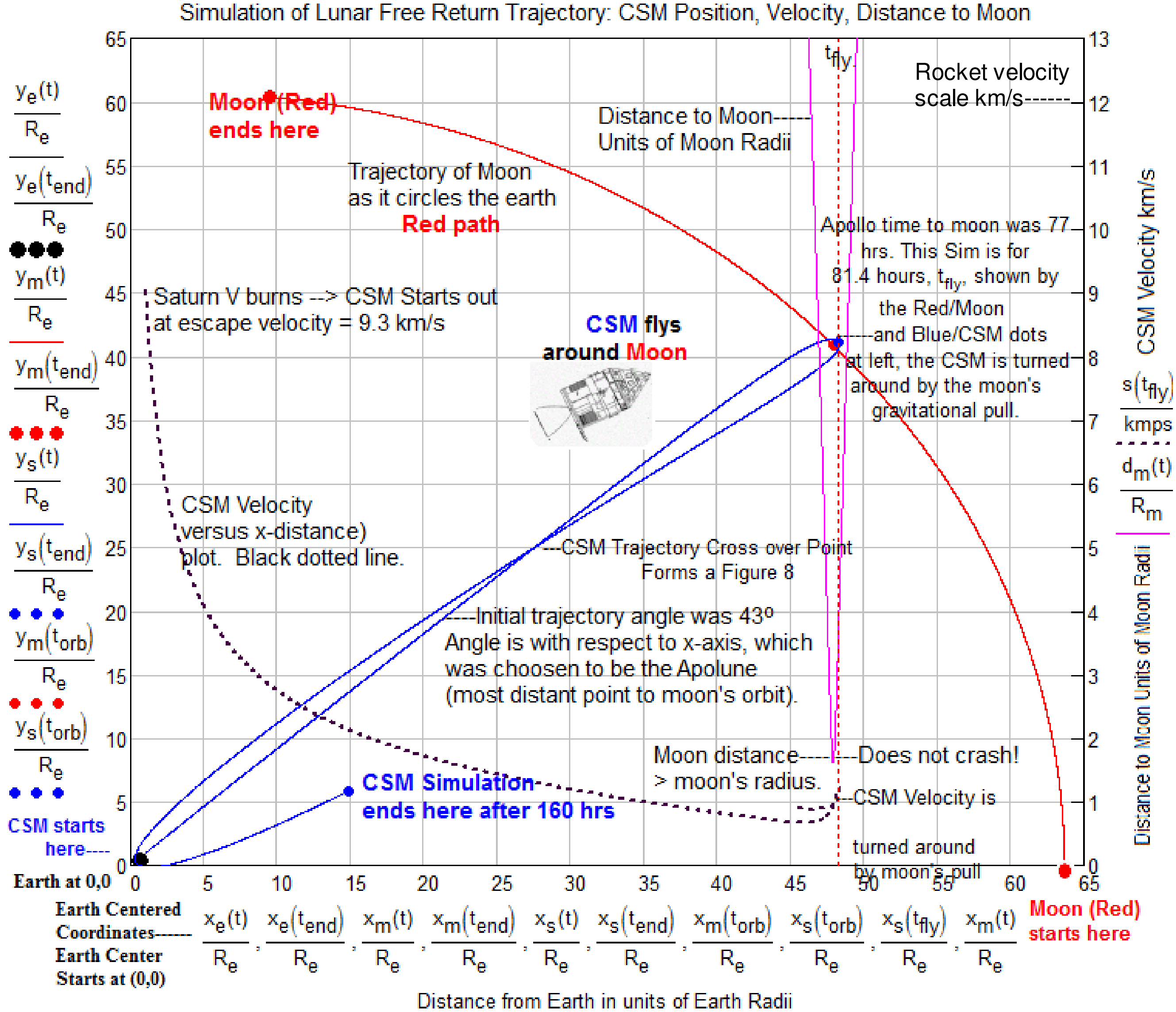
$$t := 0, \frac{t_{end}}{n_{plot}} .. t_{end} \qquad t_{fly} := 0, \frac{t_{end}}{n_{plot}} .. 90hr \qquad t_{fly.} := \frac{x_s(t_{orb})}{R_e}$$

$$vx_s(t) := \frac{d}{dt}x_s(t) \qquad vy_s(t) := \frac{d}{dt}y_s(t) \qquad s(t) := \sqrt{vx_s(t)^2 + vy_s(t)^2}$$

Finding a Free Return Trajectory (FRT) is a little tricky. First, the trajectory must catch the moon at the exact place and time as travels around the earth and then after being swing around by the moon's gravity it must swing back and catch the earth in such a way as to go into earth orbit. This can present a problem for the Differential Equation Solver. This is a three body problem. A change in the CSM's trajectory is influenced by the pull the moon, which in turn is affected by the pull of the earth. The solver can easily fail to converge on a solution. A change in angle by 10 degrees can result in a large change in orbit time of 4.5 days. We also must check that CSM does not crash into moon.

Below is a plot of our FRT solution for the Apollo Trajectory. It shows the CSM's x,y position and velocity from earth to moon and back. .Note the figure 8 orbit of this Free Return. The Apollo 11 flight time to the moon was 77 hours. Our simulation is for 81.4 hours. Because of instabilities, convergence problems, etc. some trial and error was required.

Simulation of Lunar Free Return Trajectory: CSM Position, Velocity, Distance to Moon



IB. 4-Body Sim of Apollo Free Return Trajectory: CSM to Moon & Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w **Earth at Center**

This Simulation Uses the Mathcad Differential Equation Solving Methodology discussed in: arXiv:1504.07964
"***Motion of the planets: the calculation and visualization in Mathcad***", Valery Ochkov, Katarina Pisačić

4-Body Reference Frame: Earth and moon are initially at 0,0 and the earth and sun are initially not moving.

$kg := 1$ $\quad m := 1$ $\quad s := 1$ $\quad N := 1$ $\quad s := 1$ $\quad min := 60s$ $\quad hr := 3600s$ $\quad kgf := 9.80665N$

$km := 1000m$ $\quad kmps := km$ $\quad kph := \frac{km}{hr}$ $\quad mph := 0.447 \cdot 10^{-3} kmps$ $\quad n_{plot} := 10000$

Run Simulation for 115 hrs $\quad$ **Apollo 11 Orbit 77 hr**

$FRAME := 999$ $\quad n_{ode} := 20000$ $\quad n := 999$ $\quad t_{end} := \frac{114.5hr}{n+1} \cdot (FRAME + 1)$ $\quad t_{orb} := 58.5hr$

Time of Flight (TOF) = t_{orb}

$G := 6.67384 \cdot 10^{-11} \frac{N \cdot m^2}{kg}$

Trajectory to Moon's Sphere of Influence $\quad$ Apolune

Initial x,y Velocity CSM $\quad$ **Radius of Earth** $\quad$ **Apogee to Moon**

$v_{0x} := 7.58kmps$ $\quad v_{0y} := 5.5kmps$ $\quad R_m := 1737.4km$ $\quad R_e := 6370km$ $\quad d_{m\_ap} := 405500km$

$t_{end} = 114.5 hr$ $\quad v_{CSM} := \sqrt{v_{0x}^2 + v_{0y}^2}$ $\quad v_{CSM} = 9.365 \cdot kmps$ $\quad d_{e\_ap} := 152 \cdot 10^6 km$

Define Gravitational and Dynamics Equations for Earth, Moon, and CSM

e is Earth

a is Sun

m is Moon

s is CSM

$$\begin{pmatrix} m_e & x_{e0} & y_{e0} & vx_{e0} & vy_{e0} \\ m_a & x_{a0} & y_{a0} & vx_{a0} & vy_{a0} \\ m_m & x_{m0} & y_{m0} & vx_{m0} & vy_{m0} \\ m_s & x_{s0} & y_{s0} & vx_{s0} & vy_{s0} \end{pmatrix} := \begin{pmatrix} 5.972 \cdot 10^{24} kg & 0m & 0m & 0kph & 0kmps \\ 1.989 \cdot 10^{30} kg & -130 \cdot 10^6 km & -80 \cdot 10^6 km & 0kmps & 0kmps \\ 7.347 \cdot 10^{22} kg & d_{m\_ap} & 0km & 0kmps & 0.97kmps \\ 13600kg & R_e + 110km & R_e - 96km & v_{0x} & v_{0y} \end{pmatrix}$$

Given

Set of Differential Guidance Equations for 4 Body Problem of Earth, Moon, and CSM

$x_e(0) = x_{e0}$ $\quad x_e'(0) = vx_{e0}$ $\quad y_e(0) = y_{e0}$ $\quad y_e'(0) = vy_{e0}$ $\quad x_m(0) = x_{m0}$ $\quad x_m'(0) = vx_{m0}$ $\quad y_m(0) = y_{m0}$ $\quad y_m'(0) = vy_{m0}$

$$m_e \cdot x_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (x_m(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (x_s(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (x_s(t) - x_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3}$$

$$m_e \cdot y_e''(t) = \frac{G \cdot m_e \cdot m_m \cdot (y_m(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_m(t))^2 + (y_e(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (y_s(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_e \cdot m_s \cdot (y_s(t) - y_e(t))}{\left[\sqrt{(x_e(t) - x_s(t))^2 + (y_e(t) - y_s(t))^2}\right]^3}$$

$$m_m \cdot x_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (x_e(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_s \cdot (x_s(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G \cdot m_m \cdot m_s \cdot (x_s(t) - x_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]}$$

$$m_m \cdot y_m''(t) = \frac{G \cdot m_m \cdot m_e \cdot (y_e(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]^3} + \frac{G m_m \cdot m_s \cdot (y_s(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]^3} + \frac{G m_m \cdot m_s \cdot (y_s(t) - y_m(t))}{\left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right]}$$

$x_s(0) = x_{s0}$ $\quad x_s'(0) = vx_{s0}$ $\quad y_s(0) = y_{s0}$ $\quad y_s'(0) = vy_{s0}$ $\quad x_S(0) = x_{S0}$ $\quad x_S'(0) = vx_{S0}$ $\quad y_S(0) = y_{S0}$ $\quad y_S'(0) = vy_{S($

$$m_s \cdot x_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (x_e(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (x_m(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_S \cdot (x_S(t) - x_s(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_s \cdot y_s''(t) = \frac{G \cdot m_s \cdot m_e \cdot (y_e(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_e(t))^2 + (y_s(t) - y_e(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_m \cdot (y_m(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_m(t))^2 + (y_s(t) - y_m(t))^2}\right]^3} + \frac{G \cdot m_s \cdot m_S \cdot (y_S(t) - y_s(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_S \cdot x_S''(t) = \frac{G \cdot m_S \cdot m_e \cdot (x_e(t) - x_S(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_m \cdot (x_m(t) - x_S(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_s \cdot (x_s(t) - x_S(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

$$m_S \cdot y_S''(t) = \frac{G \cdot m_S \cdot m_e \cdot (y_e(t) - y_S(t))}{\left[\sqrt{(x_e(t) - x_S(t))^2 + (y_e(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_m \cdot (y_m(t) - y_S(t))}{\left[\sqrt{(x_m(t) - x_S(t))^2 + (y_m(t) - y_S(t))^2}\right]^3} + \frac{G \cdot m_S \cdot m_s \cdot (y_s(t) - y_S(t))}{\left[\sqrt{(x_s(t) - x_S(t))^2 + (y_s(t) - y_S(t))^2}\right]^3}$$

IB. 4-Body Sim of Apollo Free Return Trajectory: CSM to Moon and Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w **Earth at Center**

Plot for Sim of 4-Body Free Return Traj: CSM to Moon and Back

Differential Equation Solver

$$\begin{matrix}\text{Earth}\\ \\ \text{Moon}\\ \\ \text{Space Craft}\\ \\ \text{Sun}\\ \\ \end{matrix}\quad \begin{pmatrix} x_e\\ y_e\\ x_m\\ y_m\\ x_s\\ y_s\\ x_a\\ y_a \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} x_e\\ y_e\\ x_m\\ y_m\\ x_s\\ y_s\\ x_a\\ y_a \end{pmatrix}, t, t_{end}, n_{ode}\right]$$

Initial Velocity (km/s) of CSM at an Altitude of 141 km:

$$\sqrt{v_{0x}{}^2 + v_{0y}{}^2} = 9.365\cdot\text{kmps}$$

time t_{fly}, just beyond lunar fly by time **t $_{flyby\ dot}$**

$$t := 0, \frac{t_{end}}{n_{plot}} .. t_{end} \qquad t_{fly} := 0, \frac{t_{end}}{n_{plot}} .. 160hr \qquad t_{fly.} := \frac{x_s(t_{orb})}{R_e}$$

$$vx_s(t) := \frac{d}{dt}x_s(t) \qquad vy_s(t) := \frac{d}{dt}y_s(t) \qquad s(t) := \sqrt{vx_s(t)^2 + vy_s(t)^2}$$

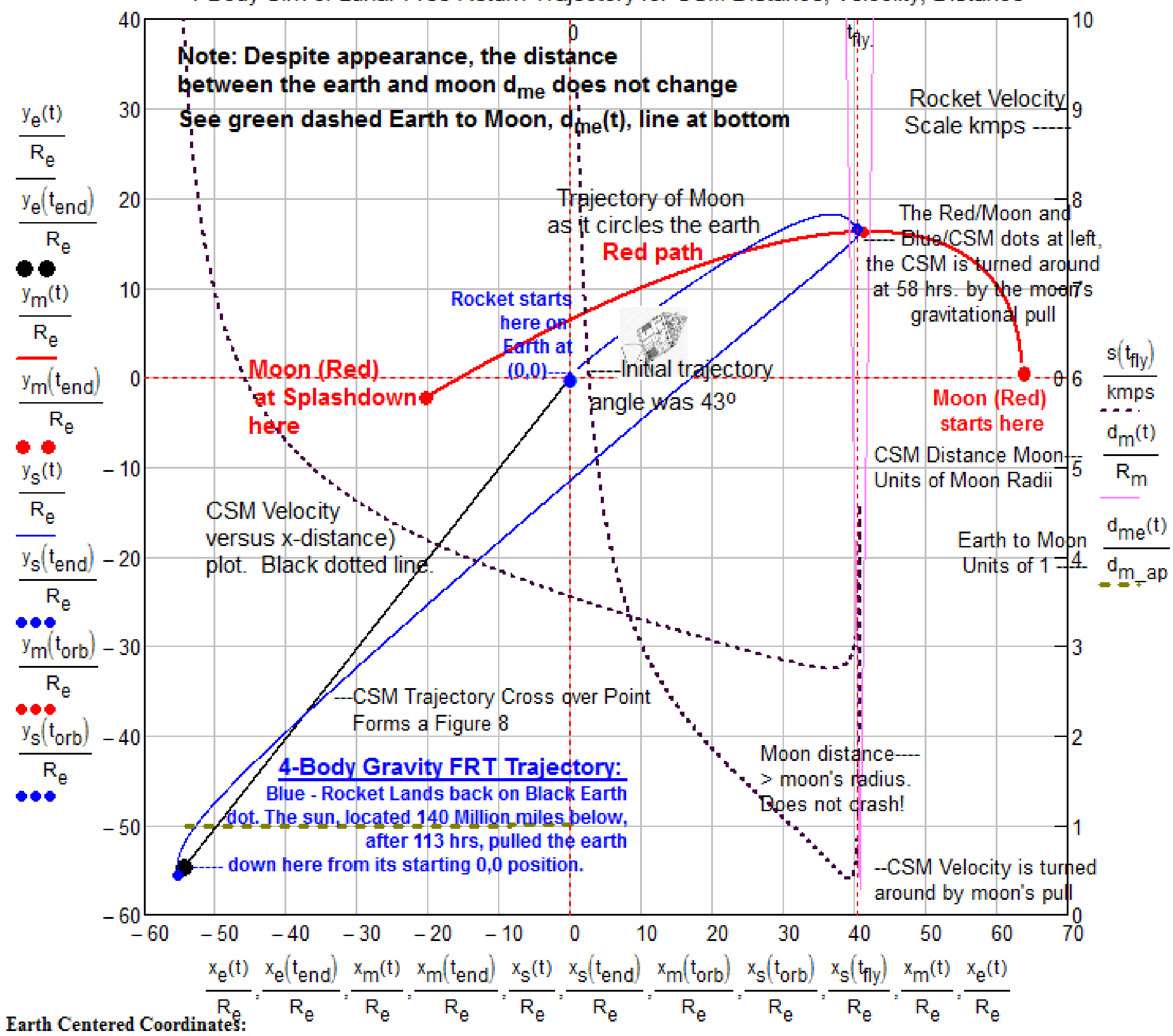
Distance from Earth to Moon

$$d_{me}(t) := \left[\sqrt{(x_m(t) - x_e(t))^2 + (y_m(t) - y_e(t))^2}\right]$$

Distance to the Center of the Moon

$$d_m(t) := \left[\sqrt{(x_m(t) - x_s(t))^2 + (y_m(t) - y_s(t))^2}\right] \qquad \frac{d_m(t_{orb})}{R_m} = 3.242$$

4-Body Sim of Lunar Free Return Trajectory for CSM Distance, Velociity, Distance



Earth Centered Coordinates: Center of Earth Starts at (0,0), but gravitational pull of sun, 94 million miles below-left of earth pulls the earth down & left from 0,0 so it ends at black dot above. The rocket (blue dot)lands back on earth 114 hours after launch.

Note: The radial velocity of the earth around the sun is 1º every 365 days or 1/365º per day. Our sim runs 114 hrs or 114/24 days. This results in (1/365º) x 114/24 or 7.5º. For the purpose of our illustration, we will ignore this added complexity. Think of this as a rotating reference frame, such as our experience of us living on a rotating earth

IC. 4-Body Sim of Apollo Free Return Trajectory: CSM to Moon and Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w **Sun at Center**

We examine 2 different versions of 4-Body Trajector & 2 Conditions of Moon Gravity: Turned Off and Turned (

In the first version, above, we put the earth at the center (0,0) and the sun is at 45° below (below left), 93 million miles aw
In the second below, we have the sun initially directly below the earth, 93 million miles away. The sun is at the center (0,(
The earth revolves around the sun with a velocity of 30 km/s and the moon revolves around the earth at 0.94 km/s. The gravitational pull of the sun pulls both the earth and moon downwards toward the sun. We solve the system of differentia equations for the spacecraft, earth, moon, and Sun. Then we do a change of variable to make the earth at the center. **Putting the sun at the center gives a more accurate simulation**. We have a far closer match to Apollo mission times. Apollo Times to Moon and Back to Earth: 77 and 142 hrs. Simulation Times to Moon and Earth: 77 and 138 hrs.

Run Simulation for 180 hrs **Apollo 11 Orbit 77 hrs**

$FRAME := 999 \quad n_{ode} := 20000 \quad n := 999 \quad n_{plot} := 10000 \quad t_{end} := \frac{138hr}{n+1}\cdot(FRAME+1) \quad t_{orb} := 77hr$

Time of Flight (TOF) = t_{orb}

$G := 6.67384\cdot10^{-11}\cdot\frac{N\cdot m^2}{kg}$

Trajectory to Moon's Sphere of Influence

Initial x,y Velocity CSM

Radius of Earth **Apogee, Perogee to Moon**

$v_{0x} := -0.0kmps \quad v_{0y} := 9.234kmps \quad R_m := 1737.4km \quad R_e := 6370km \quad d_{m\_ap} := 405500km$

$v_{CSM} := \sqrt{v_{0x}^2 + v_{0y}^2} \quad 8v_{CSM} \quad 9.229\ kmps \quad d_{e\_ap} := 147.1\cdot10^6 km \quad d_{m\_pe} := 3.45\cdot10^8 m$

Define Gravitational and Dynamics Equations for Earth, Moon, Sun and Spacecraft (CSM)

e is Earth
S is Sun
m is Moon
s is CSM

$$\begin{pmatrix} m_e & x_{e0} & y_{e0} & vx_{e0} & vy_{e0} \\ m_S & x_{S0} & y_{S0} & vx_{S0} & vy_{S0} \\ m_m & x_{m0} & y_{m0} & vx_{m0} & vy_{m0} \\ m_s & x_{s0} & y_{s0} & vx_{s0} & vy_{s0} \end{pmatrix} := \begin{pmatrix} 5.972\cdot10^{24}kg & 0m & d_{e\_ap} & -29kmps & 0kmps \\ 1.989\cdot10^{30}kg & 0m & 0m & 0kmps & 0kmps \\ 7.347\cdot10^{22}kg & 0km & d_{e\_ap}+d_{m\_ap} & -29.983kmps & 0kmps \\ 13600kg & R_e + 334.4km & d_{e\_ap}+R_e-90km & v_{0x}-29kmps & v_{0y} \end{pmatrix}$$

Given **Set of Differential Guidance Equations for 4 Body Problem of Earth, Moon, Sun, and CSM**

$x_e(0) = x_{e0} \quad x_e'(0) = vx_{e0} \quad y_e(0) = y_{e0} \quad y_e'(0) = vy_{e0} \quad x_m(0) = x_{m0} \quad x_m'(0) = vx_{m0} \quad y_m(0) = y_{m0} \quad y_m'(0) = vy_{m0}$

$$m_e\cdot x_e''(t) = \frac{G\cdot m_e\cdot m_m\cdot(x_m(t)-x_e(t))}{\left[\sqrt{(x_e(t)-x_m(t))^2+(y_e(t)-y_m(t))^2}\right]^3} + \frac{G\cdot m_e\cdot m_s\cdot(x_s(t)-x_e(t))}{\left[\sqrt{(x_e(t)-x_s(t))^2+(y_e(t)-y_s(t))^2}\right]^3} + \frac{G\cdot m_e\cdot m_S\cdot(x_S(t)-x_e(t))}{\left[\sqrt{(x_e(t)-x_S(t))^2+(y_e(t)-y_S(t))^2}\right]^3}$$

$$m_e\cdot y_e''(t) = \frac{G\cdot m_e\cdot m_m\cdot(y_m(t)-y_e(t))}{\left[\sqrt{(x_e(t)-x_m(t))^2+(y_e(t)-y_m(t))^2}\right]^3} + \frac{G\cdot m_e\cdot m_s\cdot(y_s(t)-y_e(t))}{\left[\sqrt{(x_e(t)-x_s(t))^2+(y_e(t)-y_s(t))^2}\right]^3} + \frac{G\cdot m_e\cdot m_S\cdot(y_S(t)-y_e(t))}{\left[\sqrt{(x_e(t)-x_S(t))^2+(y_e(t)-y_S(t))^2}\right]^3}$$

$$m_m\cdot x_m''(t) = \frac{G\cdot m_m\cdot m_e\cdot(x_e(t)-x_m(t))}{\left[\sqrt{(x_m(t)-x_e(t))^2+(y_m(t)-y_e(t))^2}\right]^3} + \frac{G\cdot m_m\cdot m_s\cdot(x_s(t)-x_m(t))}{\left[\sqrt{(x_m(t)-x_s(t))^2+(y_m(t)-y_s(t))^2}\right]^3} + \frac{G\cdot m_m\cdot m_S\cdot(x_S(t)-x_m(t))}{\left[\sqrt{(x_m(t)-x_S(t))^2+(y_m(t)-y_S(t))^2}\right]^3}$$

$$m_m\cdot y_m''(t) = \frac{G\cdot m_m\cdot m_e\cdot(y_e(t)-y_m(t))}{\left[\sqrt{(x_m(t)-x_e(t))^2+(y_m(t)-y_e(t))^2}\right]^3} + \frac{G m_m\cdot m_s\cdot(y_s(t)-y_m(t))}{\left[\sqrt{(x_m(t)-x_s(t))^2+(y_m(t)-y_s(t))^2}\right]^3} + \frac{G m_m\cdot m_S\cdot(y_S(t)-y_m(t))}{\left[\sqrt{(x_m(t)-x_S(t))^2+(y_m(t)-y_S(t))^2}\right]^3}$$

$x_s(0) = x_{s0} \quad x_s'(0) = vx_{s0} \quad y_s(0) = y_{s0} \quad y_s'(0) = vy_{s0} \quad x_S(0) = x_{S0} \quad x_S'(0) = vx_{S0} \quad y_S(0) = y_{S0} \quad y_S'(0) = vy_{S0}$

$$m_s\cdot x_s''(t) = \frac{G\cdot m_s\cdot m_e\cdot(x_e(t)-x_s(t))}{\left[\sqrt{(x_s(t)-x_e(t))^2+(y_s(t)-y_e(t))^2}\right]^3} + \frac{G\cdot m_s\cdot m_m\cdot(x_m(t)-x_s(t))}{\left[\sqrt{(x_s(t)-x_m(t))^2+(y_s(t)-y_m(t))^2}\right]^3} + \frac{G\cdot m_s\cdot m_S\cdot(x_S(t)-x_s(t))}{\left[\sqrt{(x_s(t)-x_S(t))^2+(y_s(t)-y_S(t))^2}\right]^3}$$

$$m_s\cdot y_s''(t) = \frac{G\cdot m_s\cdot m_e\cdot(y_e(t)-y_s(t))}{\left[\sqrt{(x_s(t)-x_e(t))^2+(y_s(t)-y_e(t))^2}\right]^3} + \frac{G\cdot m_s\cdot m_m\cdot(y_m(t)-y_s(t))}{\left[\sqrt{(x_s(t)-x_m(t))^2+(y_s(t)-y_m(t))^2}\right]^3} + \frac{G\cdot m_s\cdot m_S\cdot(y_S(t)-y_s(t))}{\left[\sqrt{(x_s(t)-x_S(t))^2+(y_s(t)-y_S(t))^2}\right]^3}$$

$$m_S\cdot x_S''(t) = \frac{G\cdot m_S\cdot m_e\cdot(x_e(t)-x_S(t))}{\left[\sqrt{(x_e(t)-x_S(t))^2+(y_e(t)-y_S(t))^2}\right]^3} + \frac{G\cdot m_S\cdot m_m\cdot(x_m(t)-x_S(t))}{\left[\sqrt{(x_m(t)-x_S(t))^2+(y_m(t)-y_S(t))^2}\right]^3} + \frac{G\cdot m_S\cdot m_s\cdot(x_s(t)-x_S(t))}{\left[\sqrt{(x_s(t)-x_S(t))^2+(y_s(t)-y_S(t))^2}\right]^3}$$

$$m_S\cdot y_S''(t) = \frac{G\cdot m_S\cdot m_e\cdot(y_e(t)-y_S(t))}{\left[\sqrt{(x_e(t)-x_S(t))^2+(y_e(t)-y_S(t))^2}\right]^3} + \frac{G\cdot m_S\cdot m_m\cdot(y_m(t)-y_S(t))}{\left[\sqrt{(x_m(t)-x_S(t))^2+(y_m(t)-y_S(t))^2}\right]^3} + \frac{G\cdot m_S\cdot m_s\cdot(y_s(t)-y_S(t))}{\left[\sqrt{(x_s(t)-x_S(t))^2+(y_s(t)-y_S(t))^2}\right]^3}$$

IC. 4-Body Sim of Free Return Trajectory: CSM to Moon & Back

Trajectory Model: 4-Body (Earth, Moon, Sun, Spacecraft) 2D Planar Point Mass w Sun **at Center**

Best Match to Apollo 11: Time to Moon Apollo - 77 hrs, Sim -77 hrs. To Earth Apollo -142 hrs Sim - 138 hrs

Differential Equation Solver

$$\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \\ x_S \\ y_S \end{pmatrix} := \mathrm{Odesolve}\left[\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \\ x_s \\ y_s \\ x_S \\ y_S \end{pmatrix}, t, t_{end}, n_{ode}\right]$$

Initial Velocity (km/s) of CSM at an Altitude of 141 km:

$$\sqrt{v_{0x}^2 + v_{0y}^2} = 9.234 \cdot \mathrm{kmps}$$

time t_{fly}, just beyond lunar fly by time

$$t := 0, \frac{t_{end}}{n_{plot}} .. t_{end} \qquad t_{fly} := 0, \frac{t_{end}}{n_{plot}} .. t_{end}$$

$$vx_s(t) := \frac{d}{dt}x_s(t) \qquad vy_s(t) := \frac{d}{dt}y_s(t) \qquad s(t) := \sqrt{\left(vx_s(t) + 29\mathrm{kmps}\right)^2 + vy_s(t)^2}$$

$$s(t_{orb}) = 3.005 \cdot \mathrm{kmps}$$

Distance to the Center of the Moon

$$d_{ms}(t) := \left[\sqrt{\left(x_m(t) - x_s(t)\right)^2 + \left(y_m(t) - y_s(t)\right)^2}\right] \qquad \frac{d_{ms}(t_{orb})}{R_m} = 3.616 \qquad d_s := d_{e\_ap}$$

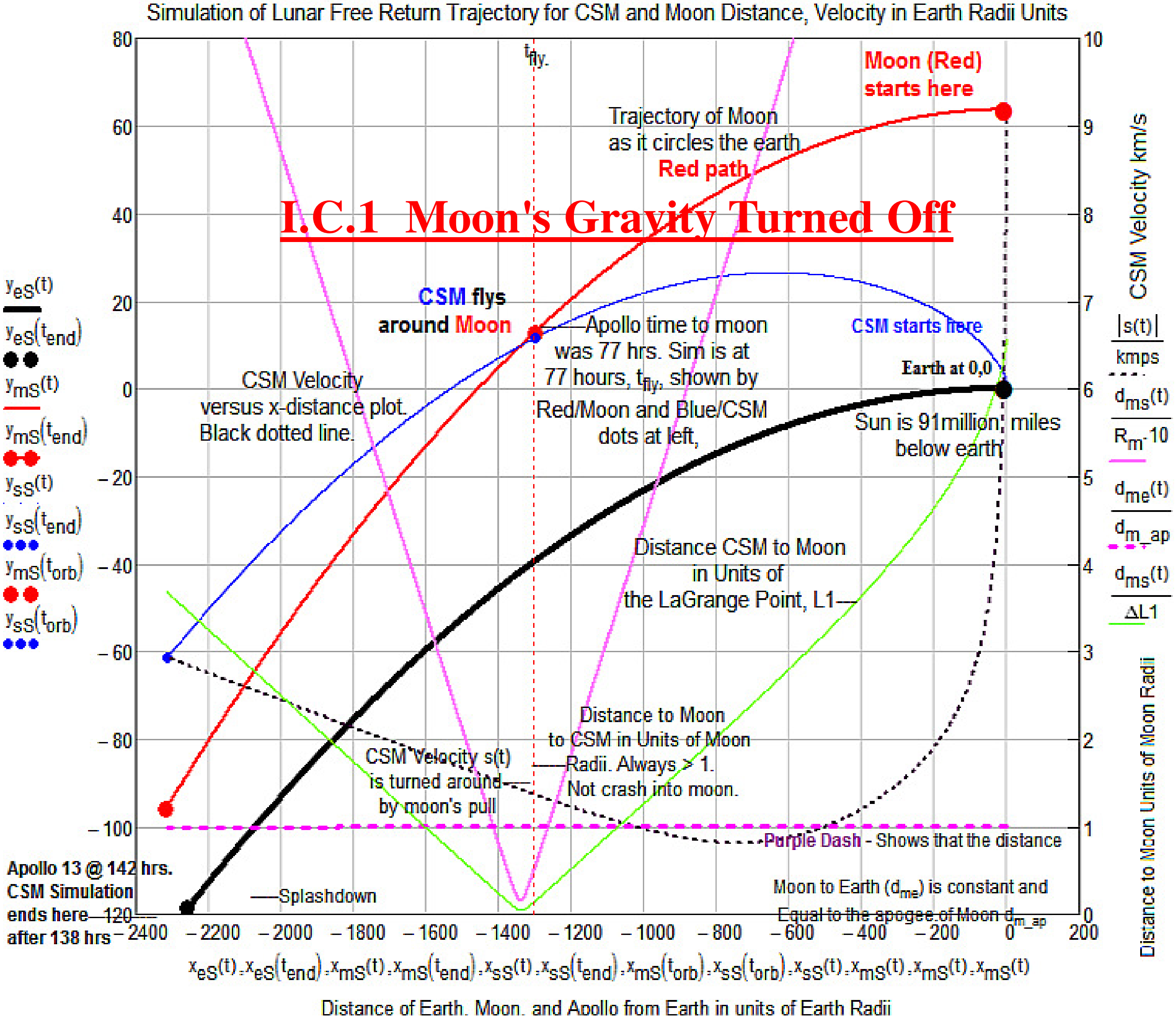
Change of Coordinates: Change to Frame Where the Earth is at Center & Use Units of Earth Radius

$$x_{eS}(t) := \frac{x_e(t)}{R_e} \qquad y_{eS}(t) := \frac{y_e(t) - d_s}{R_e} \qquad x_{mS}(t) := \frac{x_m(t)}{R_e} \qquad y_{mS}(t) := \frac{y_m(t) - d_s}{R_e} \qquad x_{sS}(t) := \frac{x_s(t)}{R_e}$$

t flyby dot

$$d_{me}(t) := \left[\sqrt{\left(x_m(t) - x_e(t)\right)^2 + \left(y_m(t) - y_e(t)\right)^2}\right] \qquad t_{fly.} := x_{sS}(t_{orb}) \qquad t_{fly.} = -1.3 \times 10^3 \qquad y_{sS}(t) := \frac{y_s(t) - d_s}{R_e}$$

Simulation of Lunar Free Return Trajectory for CSM and Moon Distance, Velocity in Earth Radii Units



I.C.2 Moon's Gravity Turned On: Free Return Trajectory

Under the influence of the moon's gravity, the CSM now loops around the moon and is redirected back to the earth on a FRT.

Note that the scale is very distorted. The x direction is 100X larger than the y direction. This is a result of the fact that the sun is located directly below and the earth and therefore the CSM are moving at 30 km/s or 67,000 mph around the sun toward the left or -x direction.

Legend - Distance (Units of Earth Radius) versus x distance

Black: Earth y Distance, $y_{eS}(t)$

Red: Moon y Distance, $y_{mS}(t)$

Blue: CSM y Distance, $y_{sS}(t)$

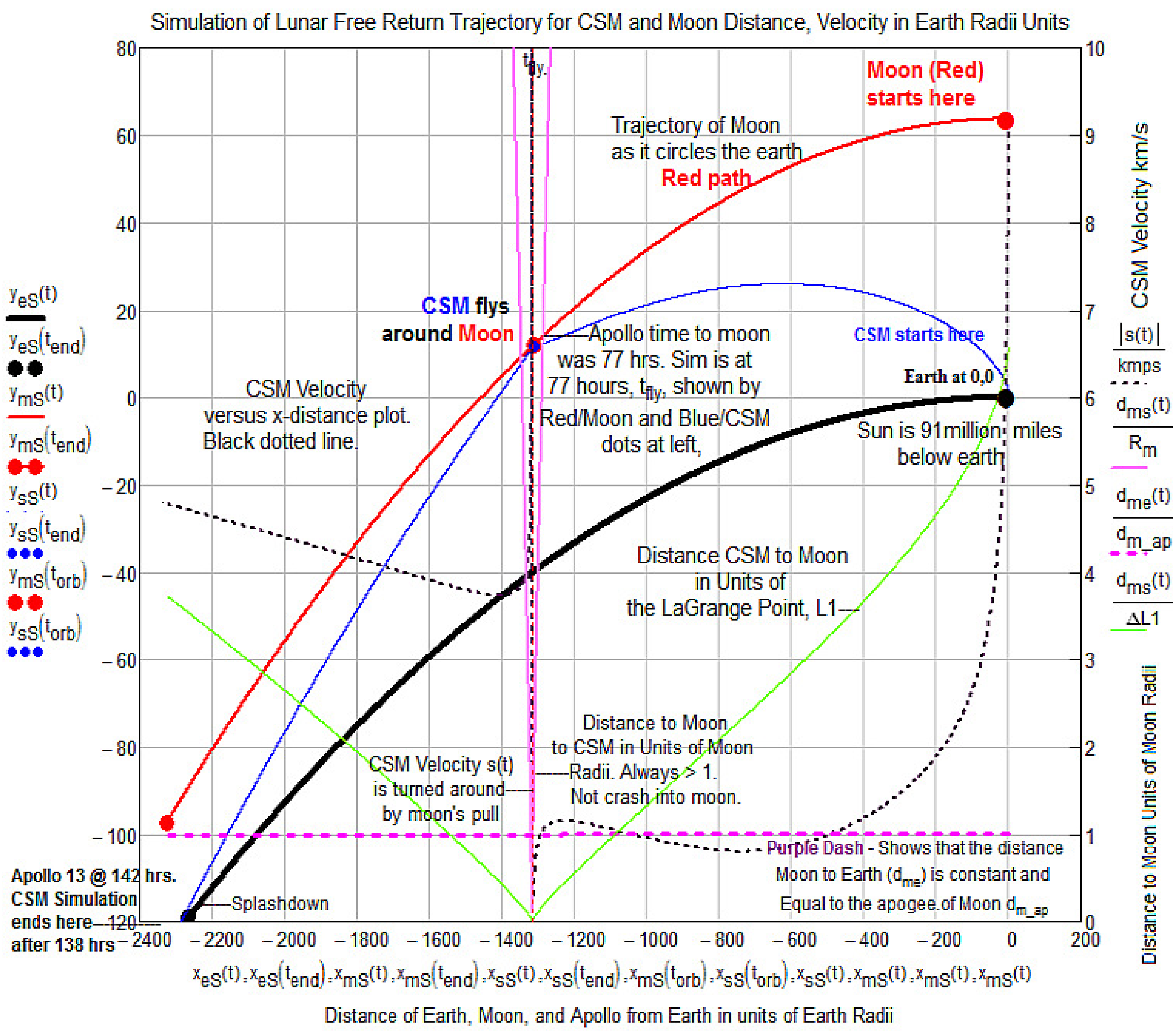
Solid Purple: CSM to Moon Distance, $d_{ms}(t)$, Units of Moon Radius

Dotted Purple: Earth to Moon Distance, $d_{me}(t)$,
Units of Distance of Earth to Moon - shows that orbit distance does not change

Dotted Green: Moon to CSM, $d_{ms}(t)$
Units of Distance to L1 Point from the Moon

Legend - Velocity

Dotted Black: CSM Speed, s(t)



II. Table of Basic Saturn IV Engine Parameters

Stages	m_{fuel}	Frame	Burn Duration		Exhaust Velocity	Burn Rate	Thrust
	($x10^6$kg)	($X10^6$kg)	(s)		(m/s)	(kg/s)	($x10^6$N)
1st stage- 5 F1s	2.04	0.136	165	Kerosene/LOX	2456	13600	33.4
2nd stage- 2 J2s	0.428	0.0432	360	LH2/LOX	4220	1190	5.02
S-IVB Orbit-1 J2	0.0356		165	LH2/LOX	4630	216	1
S-IVB TransLunar	0.0674	0.0174	312	LH2/LOX	4630	216	1
Park Orbit Payload		0.118					
Lunar Module	8200kg	2134kg		Aerozine/N2O4	3005	I_{sp}=311s	45kN
Total	2.571	0.1966					

Apollo 11 Day 1: First Stage Liftoff- Lower Atmosphere transport to 38 miles up.
The Saturn V has three states. The first Stage, which is the largest, consisted of five Rocket dyne F-1 engines, producing 33.4x 10^6 N (7.5 million pounds) of thrust. The duration of the burn is 165 seconds. The fuel was RP-1 refined kerosene, and the oxidizer was liquid oxygen (LOX). The center fifth engine was turned off early to limit the acceleration to 4 g's on the astronauts. NASA reported that the **final velocity of Stage I was of 2.76 km/s**. The spacecraft does not maintain a constant vertical launch pitch angle trajectory, but angles downward into a near horizontal orbital trajectory. The initial vertical launch had a 10 s delay, then the pitch angle decreases linearly with time. This pitch profile vs, time is simulated. See graph below. This affects vertical acceleration, avert. At the end of the first Stage the spacecraft is downrange about 58 miles (93 km) with an altitude of 38 miles. Approximate travel distance of about 69 miles.

Specific Impulse, Isp, is defined as the number of pound-s of impulse (thrust times duration) given by 1 kg mass of propellant. Kerosene (RP-1) generates an Isp in the range of 270 to 360 seconds, while liquid hydrogen (LH2) engines achieve 370 to 465 seconds.

Apollo 11 Statistics - Ground Ignition Mass: $6477875\text{lb} = 2.938\times10^{6}\,\text{kg}$

Saturn V S-I Engine Parameters:

$m_{fuel} := 2.04\cdot10^{6}\text{kg}$ $\quad m_{tot} := 2.77\cdot10^{6}\text{kg}$ $\quad m_{frame} := 1.36\cdot10^{5}\text{kg}$ $\quad R := 1.289\cdot10^{4}\cdot\frac{\text{kg}}{\text{s}}$

Engine Thrust Change in Momentum, T: T = d(mv)/dt = $v_{exhaust}$ x dm/dt = $v_{exhaust}$ x Fuel Burn Rate

$v_{exhaust} := 2715\frac{\text{m}}{\text{s}}$ $\quad v_{ex} := v_{exhaust}$ $\quad v_{ex} = 8.907\times10^{3}\cdot\frac{\text{ft}}{\text{s}}$ $\quad T := v_{ex}\cdot R = 3.5\times10^{7}\,\text{N}$

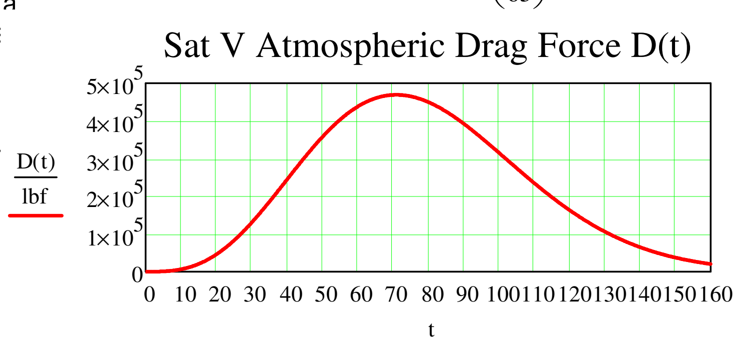
Approximate the rate of fuel consumption, mass(t), with a linear model: $\text{mass1}(t) := m_{tot} - R\cdot t$

III. Create a Model for Pitch Angle, θ of the rocket's trajectory's during Stage 1 Burn. This angle then determines the resulting g component of the earth's gravitational pull, on the rocket, **in the direction of thrust. Call this g_{thrust}.** Define a parameter, τ, that can be used to adjust the rate and final angle during burn. Adjust τ to match final velocity equal to NASA's Stage 1 velocity. Let the pitch angle, θ, decrease linearly with burn time, from initially 90 degrees vertical to ~ 30 degrees horizontal at the end of Stage 1 burn. D(t) approximates the Drag. The resultant horizontal acceleration in the direction of travel is then: a(t) - g_{thrust}

$\tau := 400\text{s}$ $\quad 90^{o} = \pi/2$ radians. $\quad \theta(t) := \frac{\pi}{2}\cdot\left[1 - \Phi(t-10\text{s})\cdot\left(0.31 + \frac{t}{\tau}\right)\right]$ $\quad S(x) := 9.027x\cdot\log\left(1+x^{2}\right)\exp\left(-x^{2}\right)$

$D(t) := 4.6\cdot10^{5}\,S\left(\frac{t}{65}\right)\cdot\text{lbf}$

IV. Saturn V Atmospheric Drag Data
The AS-503 Flight Evaluation Report gives a graph of Apollo 8 Dynamic Pressure versus **flight time**. It is approximately equal to 460,000 pounds at 70s, drops off close to zero at 160s, and has the shape shown at right." The atmosphere ends at about 100km or about 200s into the flight. Thus, we approximate the **Saturn V Atmospheric Drag Force for Apollo Missions, D(t),** as function of recorded time as shown at right.



V. Stage I Simulation Equations for Pitch, Acceleration, Velocity, and Distance

$t_{burn1} := 165s$ $\qquad g_{thrust1}(t) := g\cdot\sin(\theta(t))$

NASA's velocity was 2760 m/s

Sim Stage 1:

$$a1(t) := \frac{T - D\left(t\cdot s^{-1}\right)}{mass1(t)} \qquad v1_{thrust}(t) := \int_0^t \left(a1(t) - g_{thrust1}(t)\right) dt \qquad v1_{thrust}(165s) = 2.763\times10^3\,\frac{m}{s}$$

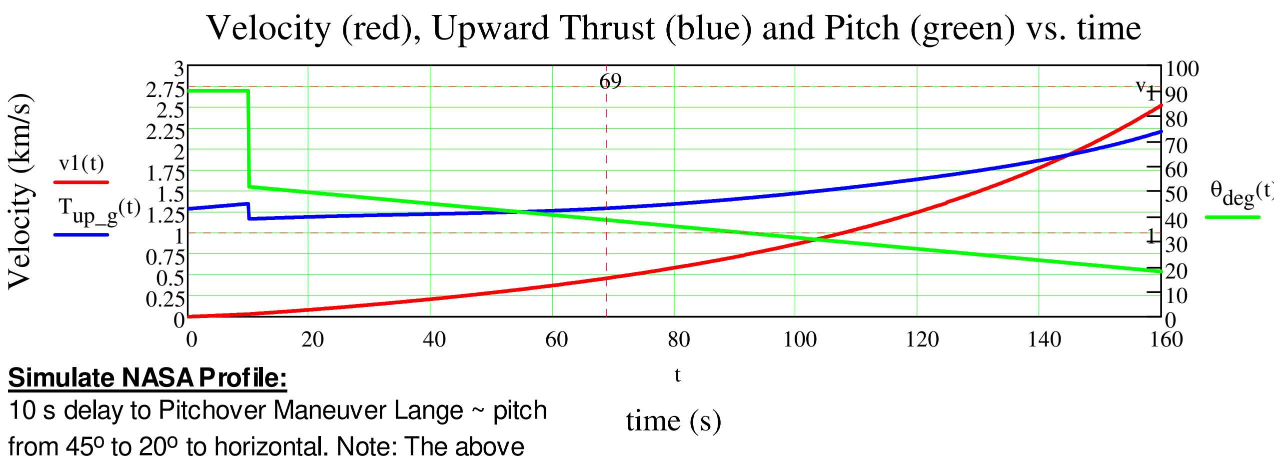
$a1(0) = 1.288\cdot g$ $\qquad a1(11s) = 1.356\cdot g$ $\qquad a1\left(t_{burn1}\right) = 5.538\cdot g$ $\qquad v1_{thrust}(69s) = 1.03\times10^3\cdot mph$

$v1_{final} := v1_{thrust}\left(t_{burn1}\right)$

VI. Graph Velocity, Vertical thrust, Tup. Check that Tup > 1 g as Pitch θ is Reduced

This is required to make certain that the vertical velocity does not decrease due to the pull of gravity

$T_{up\_g}(t) := a1(t)\cdot\sin(\theta(t))\cdot g^{-1}$ $\qquad T_{up\_g}(10.1s) = 1.168$



Simulate NASA Profile:
10 s delay to Pitchover Maneuver Lange ~ pitch from 45º to 20º to horizontal. Note: The above Thrust Angle Profile was created by approximating a graph of NASA 's pitch angle profile for State 1 launch.

VII. Calculate Stage 1 Altitude

NASA's Total Distance from launch point by Stage 1: $\quad Straight\_Line := \sqrt{(58mile)^2 + (38mile)^2} = 69.34\cdot mile$

$38mile = 61.155\cdot km$

The calculated thrust distance is 89 miles, approximately equal to NASA's Stage 1 Travel Distance of 69 miles.

$$Curved\_Path := \int_0^{t_{burn1}} \int_0^t a1(t) - g_{thrust1}(t)\,dt\,dt$$

$Curved\_Path = 87.087\cdot mile$

NASA Stage 1 altitude **38 miles.**

Calculate the Stage 1 Altitude

$$alititude := \int_0^{t_{burn1}} v1_{thrust}(t)\cdot\sin(\theta(t))\,dt \qquad alititude = 48.874\cdot mile$$

Calculate the Stage 1 Specific Impulse, I_{SP}:

Total Impulse, I $\quad I := T\cdot t_{burn1} = 5.774\times10^9\,\frac{m\cdot kg}{s}$ $\quad$ Specific Impulse, Isp $\quad I_{sp} := \frac{I}{m_{fuel}\cdot g}$ $\quad I_{sp} = 288.64\,s$

Tsiokolvsky's Equation for Stage 1 Δv: $\quad \Delta v := v_{exhaust}\cdot\ln\left(\frac{m_{tot}}{m_{tot} - m_{fuel}}\right)$ $\quad \Delta v = 3.621\times10^3\,\frac{m}{s}$

VIII. Stage IIB Burn: Velocity Calculation - Assuming 20 Deg Ascent Angle & ~ 0.35g

NASA's velocity at the end of Stage 2 was **6995 km/s** at 191 km altitude. At this altitude, drag is not significant.

$m_{fuel} := 4.28\cdot 10^{5}kg \quad m_{tot2} := 5.94\cdot 10^{5}kg \quad v_{ex2} := 4220\frac{m}{s} \quad R := 1190\cdot\frac{kg}{s} \quad t_{burn2} := 360s$

$\theta_{deg}(1) = 90 \quad g\cdot\sin\left(\frac{32}{90}\cdot\frac{\pi}{2}\right) = 0.53\cdot g$

$g_{thrust1}(199s) = 0.298\cdot g \quad \left(1 - \frac{199s}{\tau}\right)\cdot 90 = 45.225 \quad v_{ex2} = 1.385\times 10^{4}\cdot\frac{ft}{s} \quad T2 := v_{ex2}\cdot R = 5.022\times 10^{6}N$

$mass2(t) := m_{tot2} - R\cdot t \quad a2(t) := \frac{T2}{mass2(t)} - g_{thrust1}(191s) \quad v2(t) := v1_{final} + \int_0^t a2(t)\,dt$

$v2_{final} := v2(t_{burn2})$

$g_{thrust1}(191s) = 0.328\cdot g \quad a2(0s) = 0.534\cdot g \quad a2(t_{burn2}) = 2.765\cdot g$

Sim Stage IIB: $v2_{final} = 6.997\times 10^{3}\frac{m}{s}$

IX. Calculate the Required Velocity to go into Parking Orbit at Altitude of 191 km

$G := 6.67\cdot 10^{-11}\frac{m^3}{kg\cdot s^2} \quad M_e := 5.972\cdot 10^{24}kg \quad R_e := 6370km \quad v_{orbital} := \sqrt{G\cdot\frac{M_e}{R_e + 191km}}$

$v_{orbital} = 7.792\cdot\frac{km}{s}$

Earth's Escape Velocity at 50,000 km C- C Distance:

$v_{50Mm} := \sqrt{\frac{2G\cdot M_e}{50000km}} = 3.992\cdot\frac{km}{s}$

Earth's Escape Velocity from Surface of Earth:

$v_{escape} := \sqrt{\frac{2G\cdot M_e}{R_e}} = 11.183\cdot\frac{km}{s}$

X. Day 1: Stage IVB Burn #1 - Parking Orbit: Velocity Calculation

Stage IVB is close to the altitude and orbital velocity, where gravity, rather than pulling the spacecraft down, is providing the centripetal acceleration needed to keep the spacecraft in a curved or orbital path around the earth. This is the Apollo 11 earth parking orbit. It will make two trips in orbit around the earth before embarking towards the moon.

At an altitude of 191.2 km, Apollo 11 went into a parking orbit. The stated NASA stage 3 velocity was **7.791 km/s.**

$m_{fuel} := 3.56\cdot 10^{4}kg \quad m_{tot3A} := 1.23\cdot 10^{5}kg \quad v_{ex3} := 4630\frac{m}{s} \quad R := 216\cdot\frac{kg}{s} \quad t_{burn3A} := 150s$

$mass3A(t) := m_{tot3A} - R\cdot t \quad v_{ex3} = 1.519\times 10^{4}\cdot\frac{ft}{s} \quad T := v_{ex3}\cdot R = 1\times 10^{6}N$

$mass3A(t_{burn3A}) = 9.06\times 10^{4}kg \quad a3A(t) := \frac{T}{mass3A(t)} - g_{thrust1}(165s) \quad v3A(t) := v2_{final} + \int_0^t a3A(t)\,dt$

$v3A_{final} := v3A(t_{burn3A})$

$a3A(0s) = 0.407\cdot g \quad a3A(t_{burn3A}) = 0.703\cdot g$

NASA stage 3 velocity was: **7.791 km/s.**

$v3A_{final} = 1.743\times 10^{4}\cdot mph$

Sim Stage 3A Parking Orbit: $v3A_{final} = 7.791\cdot\frac{km}{s}$

XI. Orbital Mechanics - Estimate Velocity Required for Trans-Lunar Injection, TLI

See for example: ttp://www.braeunig.us/space/orbmech.htm#position

Stage IVB Burn: Fall to the Moon or Trans-Lunar Injection, TLI.
For the Apollo lunar missions, the re-startable J-2 engine in the third (S-IVB) stage of the Saturn V rocket was used to perform TLI. Apollo data states that the TLI burn provided 3.05 to 3.25 km/s (10,000 to 10,600 ft/s) of Δv, at which point the spacecraft was traveling at approximately **10.8 km/s** (34150 ft/s) relative to the Earth.

The final burn causes the orbit to change from circular (constant radius) to one that is elliptical. The final increase in velocity starts the TLI as the vehicle moves from the circular path around the earth to the orbit of the furthest point or the largest radius of the TLI. The Apollo 11 trip to the moon took 51 hours and 49 minutes. The average distance to the moon is 238,855 miles (384,400 km). To achieve orbital transfer to the moon, the vehicle must move within the **Lagrangian Point**, where the gravitational pull of the earth is equal to that of the moon. This distance from the earth is 326,054 km.
Refer to the Figure of the trajectory on page 6.

Flight Data for Apollo 11 TLI: Earth Orbit Insertion Space Fixed Velocity $25568\frac{ft}{s} = 7.793\cdot\frac{km}{s}$

Sim: Apollo Data TLI Space Fixed Velocity: $35545\frac{ft}{s} = 10.834\cdot\frac{km}{s}$

Estimate the Minimum Velocity Required for Orbit around the Moon
The minimum can be estimated by determining the change in Gravitational Potential Energy from a parking orbit 100 km above the earth to Lagrange Point 1 (LP1), located between the earth and moon, where the pull of the earth equals the pull of the moon. Lagrange Points are positions in space where the gravitational forces of a two body system like the Sun and the Earth produce enhanced regions of attraction and repulsion. There are 5 Earth-Moon Lagrange Points.

Distances

Earth to Moon — Earth to Lagrange Pt 1

$M_{earth} := 5.97\cdot 10^{24}kg$ $\quad M_{moon} := 7.348\cdot 10^{22}kg$ $\quad D_{EtoM} := 34400km$ $\quad d_{LP} := 326400km$

Radii of Earth and Moon

$m_{TLI} := 5.616\times 10^{6}\ kg$ $\quad R_{earth} := 6371km$ $\quad R_{moon} := 1079mile$

First, Calculate Change in Gravitational Potential Energy to move from Parking Orbit to Lagrange Point 1, ELP1

$$V_{earth} := -G\cdot m_{TLI}\cdot\left[\frac{M_{earth}}{R_{earth}} + \frac{M_{moon}}{D_{EtoM} - (R_{earth} + 100km)}\right] \qquad V_{LP1} := -G\cdot m_{TLI}\cdot\left(\frac{M_{earth}}{d_{LP}} + \frac{M_{moon}}{D_{EtoM} - d_{LP}}\right)$$

Gravitational PE from Earth Orbit to Lagrange Point 1, E_{LP1} $\quad E_{LP1} := V_{LP1} - V_{earth} = 3.452\times 10^{14}J$

From this Change in Gravitational Energy, ETP, we can get required velocity from Parking Orbit, to moon, v_{PO_Mn}.

From Conservation of Energy,
Needed Kinetic Energy = E_{LP1} $\quad \frac{1}{2}m_{TLI}\cdot v_{PO\_Mn} = E_{LP1}$ $\quad v_{PO\_Mn} := \sqrt{\frac{2\cdot E_{LP1}}{m_{TLI}}}$ $\quad v_{PO\_Mn} = 11.088\cdot\frac{km}{s}$
This gives Required velocity, v_{TLI}

XII. Day 1: Stage IVB Burn #2 - Trans-Lunar Injection, TLI. Earth Orbit to Moon

Adjust g for pitch and distance from the earth.

$$g_{thrust3B} := g\cdot\frac{R_e^{\,2}}{(R_e + 100km)^2}\sin\left(\frac{\pi}{2}\cdot\frac{1}{90}\right)$$

$Payload := 7.1\cdot 10^{4}kg$ $\quad T = 1\times 10^{6}N$

$m_{fuel} := 6.74\cdot 10^{4}kg$ $\quad m_{tot3B} := m_{fuel} + Payload$ $\quad R := 216\cdot\frac{kg}{s}$ $\quad t_{burn3B} := 312s$

$$mass3B(t) := m_{tot3B} - R\cdot t \qquad a3B(t) := \frac{T}{mass3B(t)} - g_{thrust3B} \qquad v3B(t) := v3A_{final} + \int_{0s}^{t} a3B(t)\,dt$$

$a3B(0s) = 0.72\cdot g$ $\quad a3B(t_{burn3B}) = 1.419\cdot g$

Apollo TLI Data: 10.8 km/s

$mass3B(t_{burn3B}) = 7.101\times 10^{4}kg$

Sim Stage 3B TLI: $v3B_{final} = 10.829\cdot\frac{km}{s}$

XIII. Graph the Velocity and Acceleration (gs) Profile of the Flight

$t_{burn1} = 165\,s \qquad t_{burn2} := 360s \qquad t_{burn3A} = 150\,s \qquad t_{burn3B} := 312$

$kmps := \dfrac{km}{s} \qquad v2(t) := v2(t \cdot s) \cdot kmps^{-1} \qquad v3A(t) := v3A(t \cdot s) \cdot kmps^{-1} \qquad v3B(t) := v3B(t \cdot s) \cdot kmps^{-1}$

$a1(t) := \dfrac{a1(t \cdot s)}{g} \qquad a2(t) := \dfrac{a2(t \cdot s)}{g} \qquad a3A(t) := \dfrac{a3A(t \cdot s)}{g} \qquad a3B(t) := \text{if}\left(t < 312, \dfrac{a3B(t \cdot s)}{g}, 0\right)$

$m_1(t) := mass1(t \cdot s) \cdot Mkg \qquad m_2(t) := mass2(t \cdot s) \cdot Mkg \qquad m_3(t) := mass3A(t \cdot s) \cdot Mkg \qquad m_4(t) := mass3B(t \cdot s) \cdot Mkg$

$$v_{SV}(t) := \text{if}\Big(t < 165, v1(t), \text{if}\Big(t < 525, v2(t - 165), \text{if}\Big(t < 675, v3A(t - 525), \text{if}\Big(t < 987, v3B(t - 675), 10.83\Big)\Big)\Big)\Big)$$

$$g_{SV}(t) := \text{if}\Big(t < 165, a1(t), \text{if}\Big(t < 525, a2(t - 165), \text{if}\Big(t < 675, a3A(t - 525), a3B(t - 675)\Big)\Big)\Big)$$

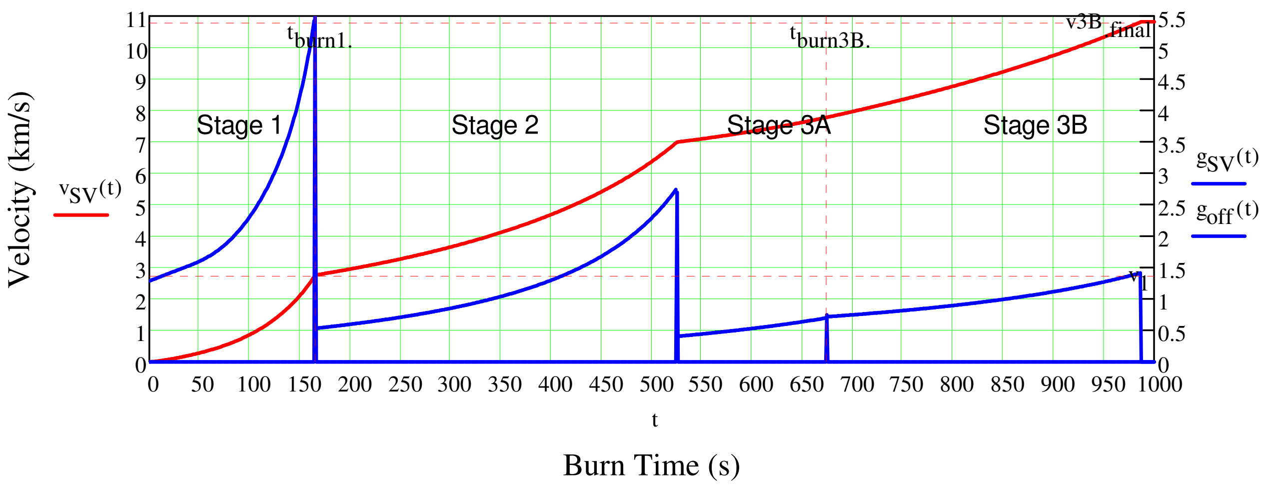
$$m_{SV}(t) := \text{if}\Big(t < 165, m_1(t), \text{if}\Big(t < 525, m_2(t - 165), \text{if}\Big(t < 675, m_3(t - 525), m_4(t - 675)\Big)\Big)\Big)$$

$t_{burn1.} := 165 \qquad t_{burn3B.} := 675 \qquad v3B_{.final} := v3B_{final} \cdot kmps^{-1}$

Note: $g_{off}(t)$ creates blue vertical markers at 165, 525, 675, and 987 s at completion of stage burns.

Simulation of the Four Saturn V Burns: Acceleration and Velocity

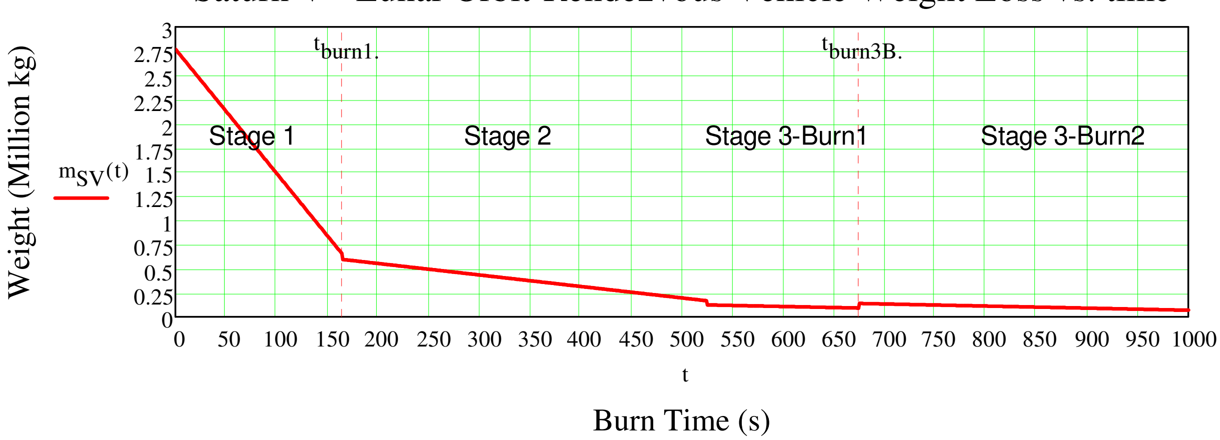
Saturn V Velocity and g Profile: Launch to Trans-Lunar Insertion



Note: The above acceleration is the net acceleration of the vehicle and not the net g thrust.
The g thrust of the engine is approximately 1 g greater than the net g force that moves the vehicle.

Sim: Weight Loss of 4 Saturn V Burns - Fuel Burns & Stage Jettisons

Saturn V - Lunar Orbit-Rendezvous Vehicle Weight Loss vs. time



The above illustrates the LOR Strategy, where a large part of the weight is jettisoned during the flight.

XIV. Day 2-4: Command/Service Module Engine for Lunar & Earth Orbits - Lunar Module

Service Module	Mass	Mass	t_{final}	Fuel	$v_{exhaust}$	Isp	$T=v_{ex}*R$
Reaction Control System, RCS	m_{fuel}	Frame	Burn Duration	(Hypergolic)	Exhaust Velocity	s	Thrust
	(kg)	(kg)	(s)		(m/s)	(s)	(kN)
Command Module		5560	Pulsed				
Service Module			Pulsed	MMH/N2O4			91
RCS	120		Pulsed	Aerozine/N2O4		290	3.87
Lunar Excursion Module							
Ascent RCS	287	4700	Pulsed	Aerozine/N2O4		311	10*410N
Ascent APS	311	2353	Pulsed	Aerozine/N2O4		311	16
LM Descent - DPS	8200		30	Aerozine/N2O4		311	45

1,050 kg Reaction Control System, RCS

CSM: Command Module, CM and Service Module, SM

The CM is the cabin that houses the three astronauts. It sits ontop of the Service Module and contains the Guidance and Navigation computer controls, which are operated by the pilot. The Command Module's Attitude Control System contains the Service Propulsion System and Reaction Control System. SM.The SM contains the SPS and RCS engines. The CM's mass is 12,250 lb (5,560 kg).
The SM consisted of twelve 93-pound-force (410 N) attitude control jets; ten were located in the aft compartment, and two pitch motors in the forward compartment. Four tanks stored 270 pounds (120 kg) of hypergolic monomethylhydrazine fuel and nitrogen tetroxide oxidizer (MMH/N2O4). The system produced small pulses or bursts thrusts as needed over a 30 minute mission period.

Service Propulsion System is the Main SM Thrust Engine

The SPS engine was used to place the Apollo spacecraft both into and out of lunar orbit, and for mid-course corrections between the Earth and Moon. It also served as a retrorocket to perform the deorbit burn for Earth orbital Apollo flights. The engine selected was the AJ10-137,[9] which used Aerozine 50 (50:50 mix by weight of hydrazine and unsymmetrical dimethylhydrazine) as fuel and nitrogen tetroxide (N2O4) as oxidizer to produce 20,500 lbf (91 kN) of thrust. It needed sufficient fuel to both get it down to the moon's surface and back up.

The Reaction Control System (RCS) provides Rotation Control in All Three Axes

The CS provides the thrust to control spacecraft rates and rotation in all three axes in addition to any minor translation maneuvers. From an entry interface of 400,000 feet, the orbiter is controlled in roll, pitch and yaw axes with the aft RCS thrusters.
The forward RCS has 14 primary and two vernier engines. The aft RCS has 12 primary and two vernier engines in each pod. The primary RCS engines provide 870 pounds of vacuum thrust each, and the vernier RCS engines provide 24 pounds of vacuum thrust each.The RCS pulses placed the CM in its proper position for re-entry into the Earth's atmosphere.
RCS propellant mass: 270 lb (120 kg) RCS engine mass: twelve x 73.3 lb (33.2 kg)

$$870\text{lbf} = 3.87 \times 10^{3}\,\text{N}$$

Lunar Excursion Module, LEM or LM: "Eagle"

LEM was the lander portion of the Apollo spacecraft built for the US Apollo program by Grumman Aircraft to carry a crew of two from lunar orbit to the surface and back. Designed for lunar orbit rendezvous, it consisted of an ascent stage and descent stage, and was ferried to lunar orbit by its companion Command/Service Module (CSM), a separate spacecraft of approximately twice its mass, which also took the astronauts home to Earth.The LEM was carried above the CM into space. After detaching and completing its mission, the LM was discarded. It was capable of operation only in outer space; structurally and aerodynamically it was incapable of flight through the Earth's atmosphere. The Lunar Module was the first manned spacecraft to operate exclusively in the airless vacuum of space. It was the first, and to date only, crewed vehicle to land anywhere beyond Earth. When ready to leave the Moon, the LM would separate the descent stage and fire the ascent engine to climb back into orbit, using the descent stage as a launch platform.

XVI Simulation of Apollo Command/Service Module (CSM) Trajectory from Earth to Moon

This Simulation Uses the Differential Equation Solving Methodology detailed in:
arXiv:1504.07964 "***Motion of the planets: the calculation and visualization in Mathcad***", Valery Ochkov, Katarina Pisačić

$kg := 1 \quad m := 1 \quad s := 1 \quad N := 1 \quad s := 1 \quad min := 60s \quad hr := 3600s \quad km := 1000m \quad kmps := km \quad kph := \frac{km}{hr}$

$deg := \frac{2\pi}{360} = 0.017 \quad FRAME := 999 \quad t_{end} := \frac{860min}{999} \cdot FRAME + 1$

Length of Simulation

$\mathbf{t} := t_{end} = 14.3336 \cdot hr \quad mph := 0.447 \cdot 10^{-3} kmps \quad kN := 1000N$

$v_0 := 1.1km \quad G := 6.67384 \cdot 10^{-11} \cdot \frac{N \cdot m^2}{kg} \quad R_m := 1737.5km \quad CSM_0 := 16500kg \quad kgf := 9.80665N$

Atmospheric Thrust, D: Time and Force

$t_1 := 677min \quad \Delta t := 6min \quad t_2 := t_1 + \Delta t \quad t_{burn} := t_1, t_1 + \frac{t_2 - t_1}{700} .. t_2 \quad D_r := 34kN \quad D(t) := if(t_1 < t < t_2, D_r, 0kgf)$

Define Gravitational and Dynamics Equations for Earth and CSM

$$\begin{pmatrix} m_m & x_{0m} & y_{0m} & x'_{0m} & y'_{0m} \\ m & x_0 & y_0 & x'_0 & y'_0 \end{pmatrix} := \begin{pmatrix} 7.35 \cdot 10^{22} kg & 0\ m & 0\ m & 0\ kph & 0\ kph \\ 16500kg & -50000\ km & R_m + 3000km & v_0 & 0kph \end{pmatrix}$$

$t_2 = 683 \cdot min \quad t_{end} = 5.16 \times 10^4 \quad CSM(t) := CSM_0 \quad D_r = 3.4 \times 10^4$

Given

$\mathbf{x}(0s) = \mathbf{x_0} \quad \mathbf{v_x}(0s) = \mathbf{x'_0}$

$\mathbf{y}(0s) = \mathbf{y_0} \quad \mathbf{v_y}(0s) = \mathbf{y'_0}$

Set of Differential Guidance Equations for CSM

$$CSM(t) \cdot \left(\frac{d}{dt} v_x(t)\right) = -G \cdot \frac{CSM(t) \cdot m_m \cdot x(t)}{\left(\sqrt{x(t)^2 + y(t)^2}\right)^3} - \frac{D(t) \cdot y(t)}{\sqrt{x(t)^2 + y(t)^2}} \qquad v_x(t) = x'(t)$$

$$CSM(t) \cdot \left(\frac{d}{dt} v_y(t)\right) = -G \cdot \frac{CSM(t) \cdot m_m \cdot y(t)}{\left(\sqrt{x(t)^2 + y(t)^2}\right)^3} + \frac{D(t) \cdot x(t)}{\sqrt{x(t)^2 + y(t)^2}} \qquad v_y(t) = y'(t)$$

Differential Equation Solver

$$\begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix} := Odesolve\left[\begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix}, t, t_{end}\right]$$

$t := 0s, \frac{t_{end}}{7000} .. t_{end} \quad t_{spdn} := 0s, \frac{t_2}{7000} .. t_2 \quad \alpha := 0, \frac{2\pi}{1000} .. 2\pi$

$t_{orb} := t_2, t_2 + \frac{t_2 - t_1}{70} .. t_{end} \quad \mathbf{D} := D(t_{end}) \quad R_{LM}(t) := \sqrt{x(t)^2 + y(t)^2}$

$$D_x(t) := \frac{D(t) \cdot y(t)}{\sqrt{x(t)^2 + y(t)^2}} \qquad D_y(t) := \frac{D(t) \cdot x(t)}{\sqrt{x(t)^2 + y(t)^2}} \qquad m_s := CSM(t_{end})$$

$Y_{dot} := \left(y(t_2 + 40s) \quad y(t_{end})\right)^T$

$F_x := -D_x(t_1 + 10s) \quad F_y := -D_y(t_1 + 10s) \quad F_y = -1.082 \times 10^4 \cdot kg \quad \mathbf{D} = 0 \cdot kgf \quad F_x = -3286.6 \cdot kgf \quad X_{dot} := \left(x(t_2 + 40s) \quad x(t_{end})\right)^T$

Simulation of Apollo Command/Service Module Trajectory from Earth to Lunar Orbit

Apollo 11 CS< approached the moon.

Descent speed (s) is 1.69 km/s. Sim is a good match.

Results of Simulation for Lunar Height and Speed

See Red Band in Below Plot for Lunar Capture where. 2 burns: Retro-thrust to slow to 2.5km/s. 1st burn:6 minutes into 315 x 111 km orbit. 2nsd:17 sec to circulize orbit to 69 miles. 2 more corrections.

$h(t) := R_{LM}(t) - R_m$

$R_{LM}(t_1) = 2376 \cdot km$ $\quad h(t_1) = 638.8 \cdot km$ $\quad h(t_2) = 483.521 \cdot km$

$R_{LM}(t_2) = 2221 \cdot km$ $\quad s(t_1) = 2.258 \cdot kmps$ $\quad s(t_2) = 1.627 \cdot kmps$

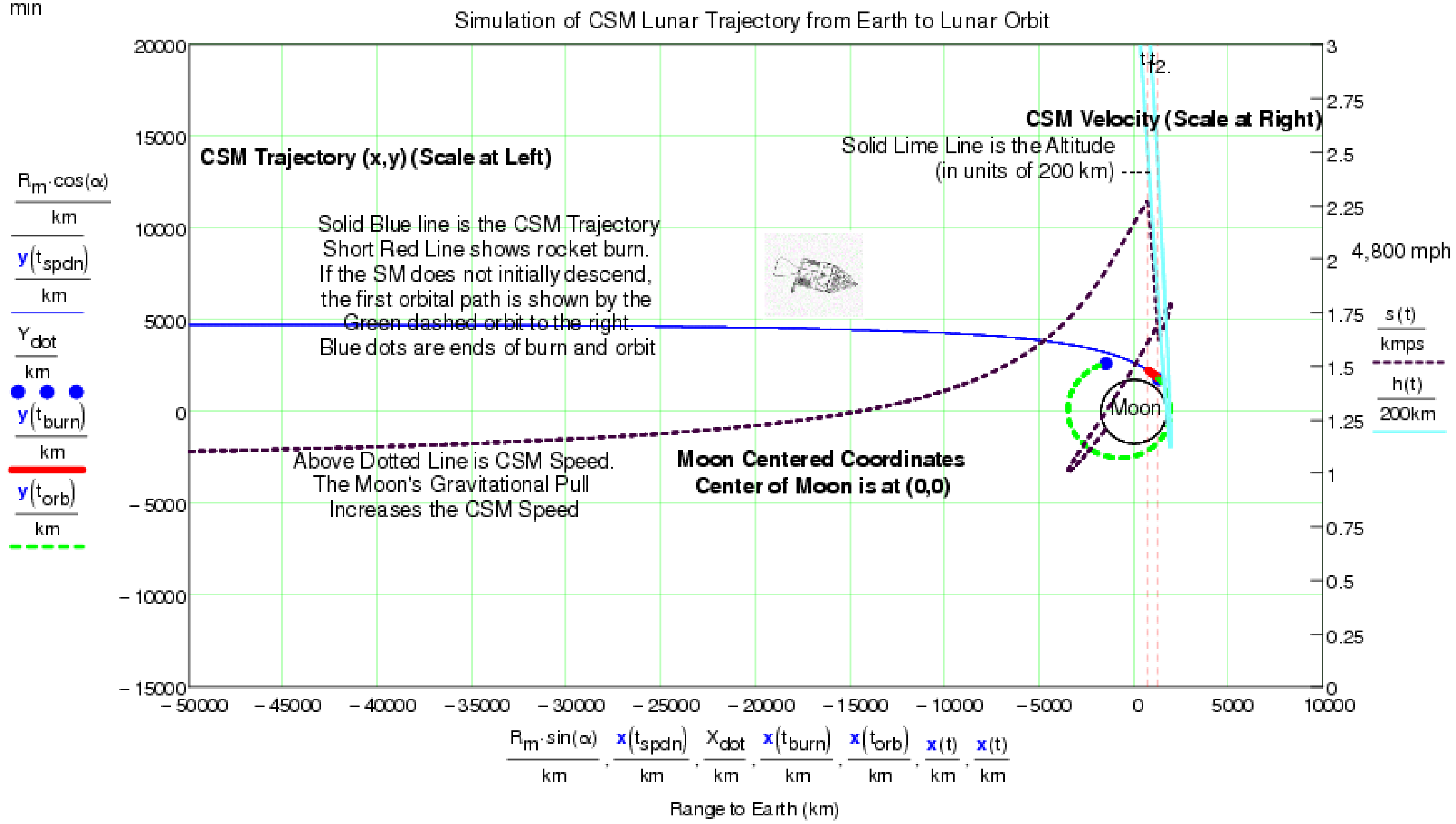
$s(t) := \sqrt{v_x(t)^2 + v_y(t)^2}$ $\quad v_{escape} = 2.4$ kmps

$s(t_2) = 1.627 \cdot kmps$ $\quad s(t_{end}) = 1.205 \cdot kmps$

$s(t_2) = 3.641 \times 10^3 \cdot mph$ $\quad s(t_{end}) = 2.695 \times 10^3 \cdot mph$

$x(t_1) = 735.841 \cdot km$ $\quad x(t_2) = 1.276 \times 10^3 \cdot km$

$\frac{t_1}{min} = 677$



XVI: Simulation of Descent from Orbit to Moon Surface, Lunar Orbit Descent, LOD

Day 4: Descent has four stages: Orbital, Braking, Approach, and finally, Hovering/Landing

Braking Phase Descent began at 48,814 feet (15 km) from a slightly elliptical coasting lunar orbit, and time to touchdown is 12 minutes. (This would require an average velocity of about 20.5 m/s or 46 mph). Descent initial orbit velocity is 1.695 km/s (5560 fps). Attitude goes from 24° to 42° in 8 minutes. Target Location is the Sea of Tranquility is centered at latitude II. 5' N and longitude 23. 5° E. **See Section XVII for Nav. and Guidance**

Descent is computer controlled. 4 radars target landing site and compute the time to go from current to desired conditions, TGO. A Quadratic Law is used to computed TGO from measured jerk, velocity, and altitude using the NAV routines. The acceleration differential between commanded and lunar acceleration is calculated. This is converted to required thrust and throttle time. When throttle time exceeds throttle region of 10 to 60%, full throttle or thrust is applied. The Ignition logic determines the time for burn to be applied. Guidance logic is used to steer craft to selected landing location. target location determines the required trajectory shaping.
Descent uses two Control Factors: Uses Two/Throttle and Pitch Angle - **See Throttle in Graph Below**

Approach Phase. During the approach phase, the altitude decreases from 7000 to 500 feet, the range decreases from approximately 4 5 nautical miles to 500 feet, and time of flight is approximately 1 minute 40 secs.

LEM (Eagle) Descent Vehicle and Engine Parameters

Mass of LEM $m_{lod} := 8200\text{kg}$

Thrust of LEM $T_{lod} := 44.5\text{kN}$

$g_{moon} := 1.62\dfrac{m}{s^2}$

Initial Orbital Velocity $v_{orb} := 1.695\dfrac{km}{s}$

Throttle Profile%: $\text{Throttle}_{des}(t) := \text{if}\left[t < 60, 0.4, \text{if}\left[t < 400, 1, \text{if}\left[t < 500, 0.6, 0.6\cdot e^{-1\cdot\left(\frac{t-500}{400}\right)}\right]\right]\right]$

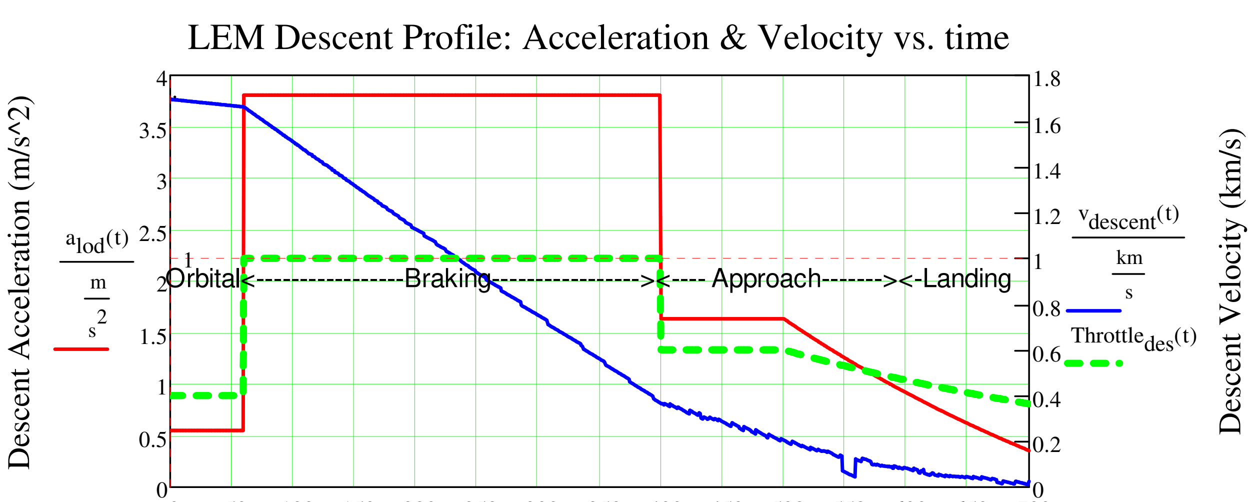
Full Throttle: $\dfrac{T_{lod}}{m_{lod}} - g_{moon} = 3.807\dfrac{m}{s^2}$

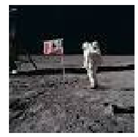
LEM Acceleration: $a_{lod}(t) := \text{Throttle}_{des}(t)\cdot\dfrac{T_{lod}}{m_{lod}} - g_{moon}$

Simulate Descent Velocity: $v_{descent}(t) := v_{orb} - \int_0^t a_{lod}(t)\,dt\cdot s$

Day 4: Simulation of LEM "Eagle" Descent to Moon Surface

Green Dotted Line -Throttle Profile
Red: Acceleration
Blue: Orbital to Descent Velocity





July 20, 1969, 20:17:14 UTC: "***Houston, Tranquillity Base here. The Eagle has landed.***"

XVII: Simulation of Ascent from Moon Surface to Orbit, Lunar Orbit Ascent, LOA

LEM Ascent Stage - Day 5 From Tranquility Base

Ascent has a single objective, namely, to achieve a satisfactory orbit from which rendezvous with the orbiting GEM can subsequently be performed. Nominally, insertion into a 9 by 45 mile at an altitude of 60,000 feet, is desired.
Ascent Engine Throttling Notes: The LMDE operates in two regimes: Full Throttle Power (FTP) is approximately 94.2% of rated thrust (9,900 lbs), and the Throttling Regime goes from 12.2% of rated thrust to 65%~ rated thrust (1,280 lbs to 6,825 lbs).

the Lunar Module Ascent stage didn't't need to attain the 2.4 km/s to escape the Moon's gravity, it just had to reach a lunar orbit (orbital speed in the range **1.5-1.7 km/s**) to rendezvous with the CSM. **Seven minutes** after ignition the astronauts would be in lunar orbit awaiting the rendezvous with the CSM. Once the LM crew transferred into the CSM, their LM Ascent stage was abandoned. All three crew returned to Earth in the Command Module.

The engines of the RCS are placed at the four corners of the lunar module because the center of gravity is shifted from the axis of the ascent module as the tanks are emptying. Thus the combined forces of the thrust and the lunar gravity create a torque which makes the ascent module turn clockwise. So that the ascent module keeps a steady direction, this torque needs to be corrected by applying a counter torque. The powered ascent is divided into two operational phases: vertical rise and orbital insertion.
5536 ft/s at 60,000 ft. The required Δv is 6056 fps. For Apollo 14 the theoretical minimum would have been 6045.3 fps, **1842.6 m/s over ~430 seconds** of flight time. After 10 sec of Thrust Angle from vertical of 0 degrees thrust angle varies linearly from 50 to 90 degree from 10 to 430 s.

Run Simulation with Variable Pitch Angle and Engine Throttled from 90 to 64% Thrust

$5536\frac{ft}{s} = 1.687\times 10^{3}\frac{m}{s}$ $\qquad \theta_{asc}(t) := \frac{\pi}{2}\cdot\left[1-\Phi(t-10)\cdot\left(0.31+\frac{t}{700}\right)\right]$ $\qquad g_{thr\_asc}(t) := g_{moon}\cdot\sin\left(\theta_{asc}(t)\right)$

$m_{loa} := 2132kg$ $\qquad T_{loa} := 16kN$ $\qquad a_{loa}(t) := \frac{T_{loa}}{m_{loa}} - g_{thr\_asc}(t)$ $\qquad a_{loa}(450) = 7.385\frac{m}{s^{2}}$

$Throttle_{asc}(t) := if(t<40, 0.9, if(t<435, 0.648, 0.75))$ $\qquad Thr_{asc}(t) := 100\,Throttle_{asc}(t)$ $3500lbf = 1.557\times 10^{4}N$

$\theta_{asc.}(t) := 90\cdot\left[1-\Phi(t-10)\cdot\left(0.31+\frac{t}{700}\right)\right]$

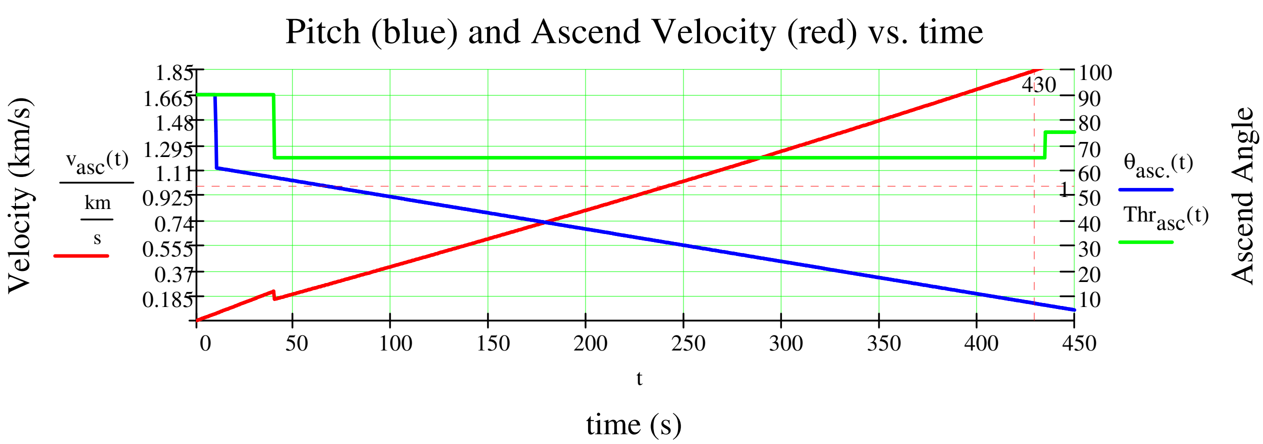
Ascent Target 1843 m/s at 430 s

Simulate Ascent Velocity: $v_{asc}(t) := Throttle_{asc}(t)\cdot\int_0^t a_{loa}(t)\,dt\cdot s$

$v_{asc}(430) = 1.844\times 10^{3}\frac{m}{s}$

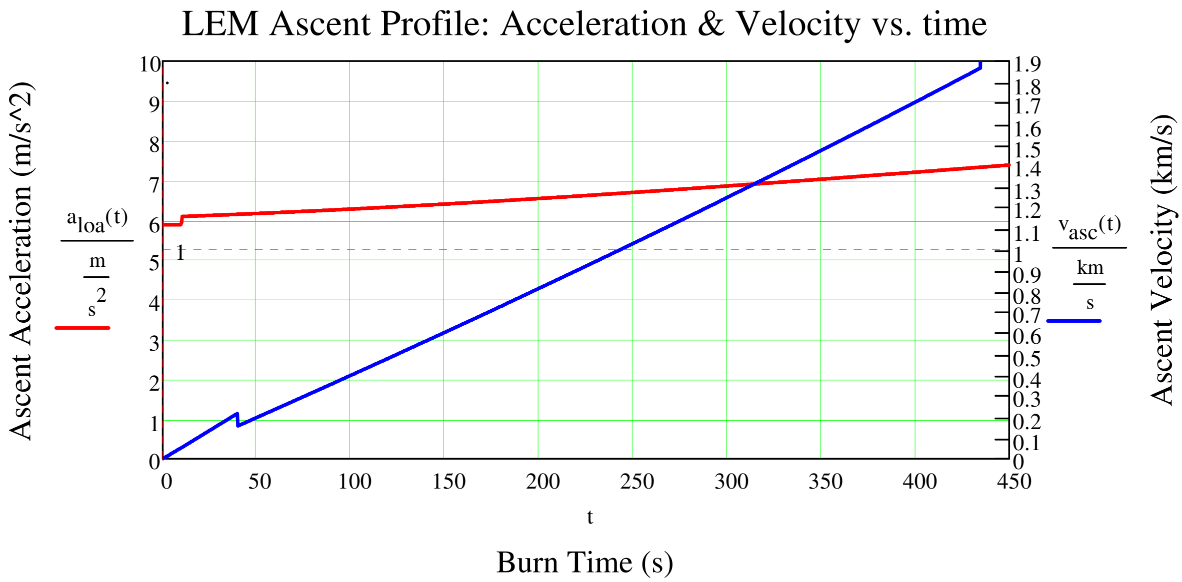
LEM Ascent to Lunar Orbit Simulation:

Blue -Thrust Angle Profile of Trajectory
Red: Ascent Velocity Rise Up to Orbital Velocity
Green: Engine Throttle



Plot of LEM's Ascent from Moon's Surface to Lunar Orbit:

Red: Ascent Acceleration
Blue: Ascent to Lunar Orbital Velocity



Day 6: Rendezvous and Docking

Eagle is now safely back in its initial lunar orbit.

https://history.nasa.gov/afj/ap11fj/19day6-rendezvs-dock.html

To rendezvous with his CSM the LEM executed a series of burns by its reaction control thrusters, controlled by the LEM computer on the basis of data supplied by Houston mission control, that initially put it in a circular orbit at an altitude of 69 miles concentric with the CSM, and then slowed it down to dock with the CSM. The LEM commander, took over control for the final docking maneuver. Return docking was very crucial and difficult. The crew returned to the command module and the hatch was sealed. The CSM and LM separated and the lunar module was jettisoned. Only the LEM ascent stage was left in lunar orbit.

XVIII: Command Service Module "Columbia" Return Flight to Earth Orbit

Trans-Earth Injection (TEI) by the SPS engine on the CSM

The Eagle reached an initial orbit of 11 by 55 miles above the moon, and when Columbia was on its 25th revolution. As the ascent stage reached apolune at 125 hours, 19 minutes, the reaction control system, or RCS, fired so as to nearly circularize the Eagle orbit at about 56 miles, some 13 miles below and slightly behind Columbia. The SPS fired for two-and-a-half minutes when Columbia was behind the moon in its 59th hour of lunar orbit.

A Trans-Earth Injection (TEI) is a propulsion maneuver used to set a spacecraft on a trajectory which will intersect the Earth's Sphere of influence, usually putting the spacecraft on a **Free Fall** return trajectory to earth. It was performed by the SPS engine on the Service Module after the undocking of the (LM) Lunar Module if provided. An Apollo TEI burn lasted approximately 203.7 seconds, providing a postgrade velocity increase of **1,076 m/s** (3,531 ft/s).

Sim: SPS engine Burn on the Service Module after the undocking from the Lunar Module (LM)

TEI Control Parameters from Influence of Moon (g-moon) to Earth (g-earth)

SM Thrust — **Throttle Factor** — **SM acceleration** — **SM mass**

$m_{SM} := 14000\text{kg}$

$T_{SM} := 91\text{kN}$

$\text{Throttle} := 0.9$

$$a_{moon} := \frac{\text{Throttle}\cdot T_{SM}}{m_{SM}} - \sin(20\text{deg})\cdot g_{moon}$$

$t_{burnSM} := 204\text{s}$

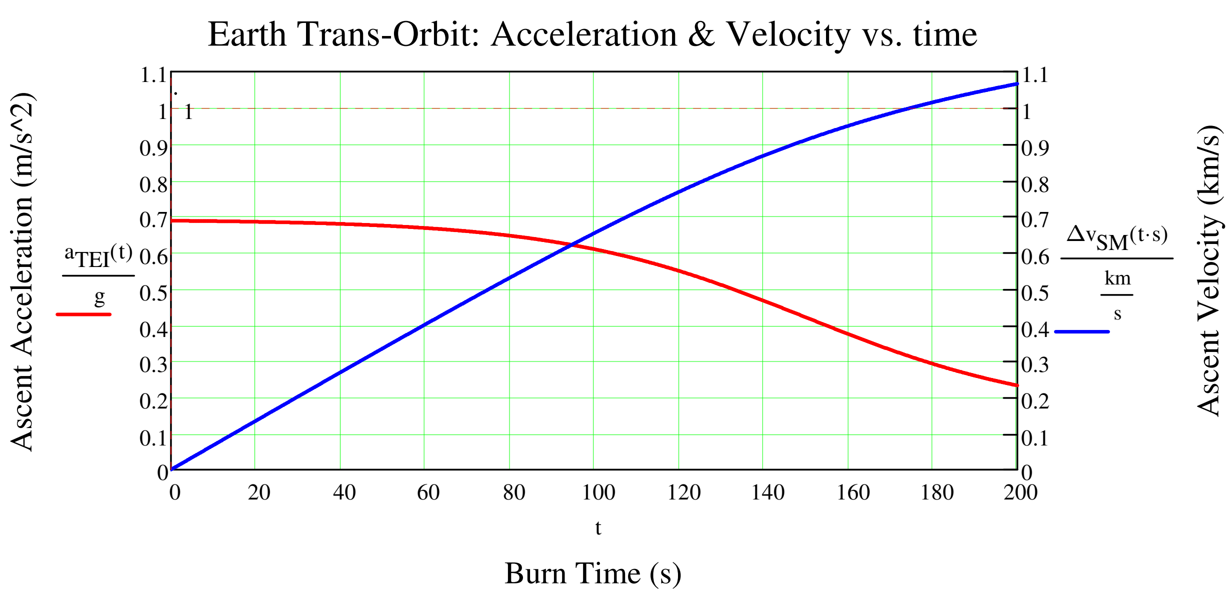
$a_{moon} = 3.269\cdot g_{moon}$

$$a_{TEI}(t) := \left[\frac{a_{moon}}{1 + e^{-\left(\frac{150-t}{29}\right)}} + .151\text{g}\right]$$

$$\Delta v_{SM}(t) := \int_0^t a_{TEI}\left(\frac{t}{s}\right)\,dt$$

Apollo Mission Data = 1.076 km/s

$$\Delta v_{SM}(t_{burnSM}) = 1.075\times 10^3\,\frac{\text{m}}{\text{s}}$$



XIX. Trans-Earth Coast, Mid-Course Correction, and CM/LEM Separation

Trans Earth Coast.
Apollo 11's return voyage to the Earth was powered by the Earth's gravity. The gravitational effects of the trans-lunar coast were reversed. The velocity of the CSM initially slowed due to the gravitational pull from the earth, but as the spacecraft moved away from the moon, the moon's gravitational effect diminished, while the Earth's gravitational pull increased. Once the earth's gravitation became dominant, the space craft was **accelerated into a freefall,** all the way to the Earth, reaching a velocity of 25,000 mph, as it entered the Earth's atmosphere over three days later.

Command Module/Service Module Separation.
Shortly before entering the Earth's atmosphere at an altitude of 400,000 feet (75 miles), the service module was jettisoned by simultaneously firing of the reaction control thrusters in both the service module and the command module. The command module is then rotated by a 180° to turn the blunt end toward the Earth.

Reentry.
Landing is just as dangerous as taking off, and ***the precise reentry trajectory is critical*** to making a safe landing. Once initiated there is ***no possibility of a second chance*** by going around and trying again if things go wrong. The initial atmospheric drag of 0.05 g experiences by the capsule as it entered the atmosphere, triggers the Earth landing subsystem, which controlled the reentry process. The Command Module entered the atmosphere, blunt end first, at 400,000 feet with a velocity approaching 25,000 mph at an angle of 6.488° to the horizontal, and flew about 1240 miles from the Earth to the designated landing point in the Pacific ocean. At the very high entry velocity, it compressed the air as the capsule heated its surface to up to around 2760° C, or 5000F, hot enough to vaporize most metals, turning the capsule into a shooting star. A 2 1/2 inch thick, sacrificial ablative heat shield, which burns and a erodes, in a the self-controlled way, carrying heat away with its combustion products, protected the capsule from the heat of reentry.

Mid-Course Correction.
Similar to the outward journey, the velocity and angle of approach to the Earth had to be very precisely controlled to ensure capture by the Earth's gravity and splashdown into the designated area

XX. Re-entry into the Earth's Atmosphere:

Challenges of High Speed Atmospheric Entry

$\text{Mega} := 10^6$

Crewed space vehicles must be slowed to subsonic speeds before parachutes or air brakes may be deployed. Such vehicles have kinetic energies typically between 50 and 1,800 megajoules (Apollo 50,000 megajoules), and atmospheric dissipation is the only way of expending the kinetic energy. The amount of rocket fuel required to slow the vehicle would be nearly equal to the amount used to accelerate it initially, and it is thus highly impractical to use retro rockets for the entire Earth re-entry procedure. While the high temperature generated at the surface of the heat shield is due to adiabatic compression, the vehicle's kinetic energy is ultimately lost to gas friction (viscosity) after the vehicle has passed by. Ballistic warheads and expendable vehicles do not require slowing at re-entry, and in fact, are made streamlined so as to maintain their speed. For Earth, atmospheric entry occurs at the Kármán line at an altitude of 100 km (62.14 mi / ~ 54 nautical mi) above the surface.

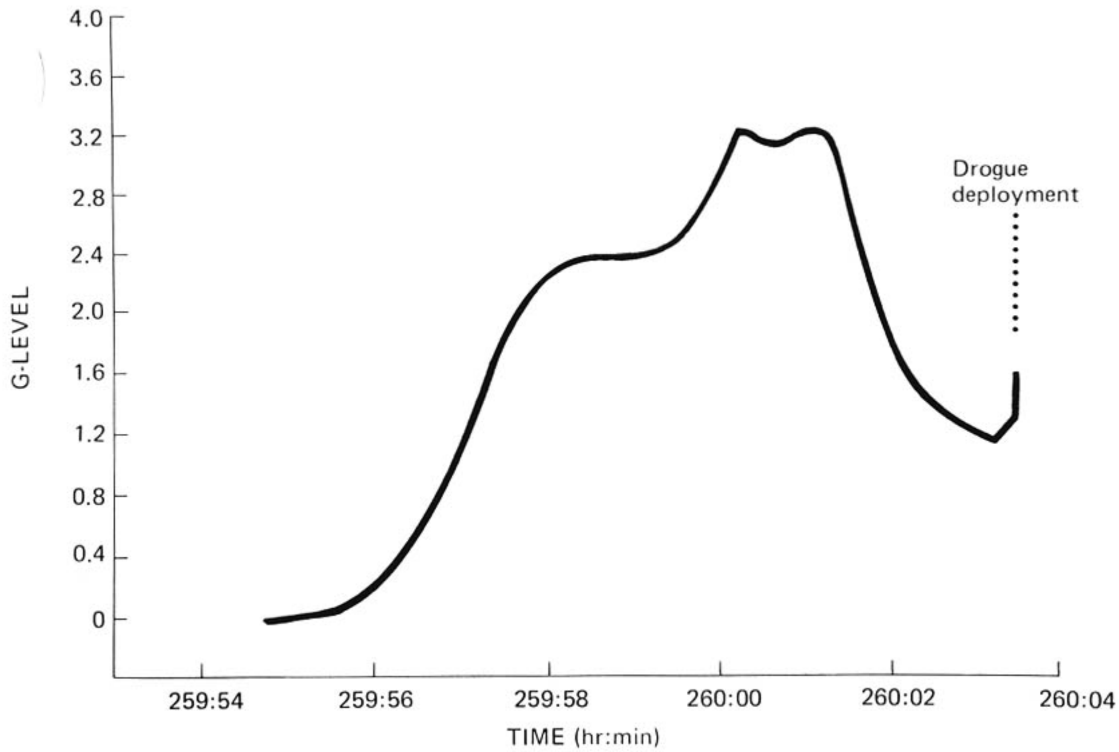
Apollo 11 Re-entry procedures were initiated July 24, 44 hours after leaving lunar orbit. The SM separated from the CM, which was re-oriented to a heat-shield-forward position.

Apollo 11 was 19,914 nautical miles [36,881 km] from the Earth approaching at a velocity of 13,695 feet per second [**4,174 m/s or Mach 12.3]**. It was 2 hours, 14 minutes, 16 seconds away from entry into the atmosphere.
The CM then adjusted its attitude -- or orientation respective to the Earth's surface -- using its thrusters so that the base of the module faced towards the Earth's surface.

Apollo Journal: https://www.hq.nasa.gov/alsj/

Earth Orbital Re-entry Atmospheric Drag Profile - Apollo 7

SP-368 Biomedical Results of Apollo



XXI. Strategies for Dissipation of Heat from Atmospheric Braking

Command Module, CM, Heat Shields

The interior of the command module must be protected from the extremes of environment that will be encountered during a mission. The heat of launch is absorbed principally through the boost protective cover, a fiberglass structure covered with cork which fits over the command module like a glove. The boost protective cover weighs about 700 pounds and varies in thickness from about 3/10 of an inch to about 7/8" (at the top).

There are 4 possible ways to dissipate or shield a rocket's interior from this heat.
1. ICBMs come in at high angles and spend little time in that atmosphere to minimize random effects of turbulence and distortions of the atmosphere and thus attain high target accuracy.The more time a missile spends in the atmosphere, the less accurate.it will be. ICBMs employ heat sinks, which delays the time it takes to spread the heat. This thermal delay to peak temperature, protects the electronics before the target hit.

2. The Space Shuttle came in at an angle and, slowed down, and used tiles that get very hot & radiated the heat away. It used 22,000 ablative insulated silica glass tiles, glued to the aluminum body with silicone adhesive, that heated up to very high temperatures, but provided thermal insulation, and then radiated the heat away. The risk of this method was that loss of just a few tiles in a critical area, left that region unprotected, and the result could be catastrophic. The result could be like a blow torch on the aluminum body. This was the cause of the Columbia disaster in 1981. It was estimated that tiles represented 10% of the risk of failure of Space Shuttle.

3. Apollo Missions used the ablation method. They used flat nose cones and a shallow entry angle to slow down entry. The slow re-entry also minimized the de-acceleration to protect the astronauts.
The flat Apollo CM nose cone heated up, but they had a ablative material deposited on the cones that absorbed and ablated the heat. This heat went into ablating (melting and boiling off) the nose cone material. The process was similar to pouring water on a fire, or using ice to cool down a drink on a hot day, only instead of melting of ice in drinks, the ablative material vaporizes. Vaporization, like boiling water, absorbs and carries huge amounts of heat away. Cone materials are used that have a very large heat of fusion. The ablative material that did this job for Apollo was a silica impregnated phenolic epoxy resin, a type of reinforced plastic.
Total weight of the shield was about **3,000 lbs. Heat of fusion** 143 kCal/mol. 143 kCal/68 gm = **8.8*10^6 J/kg**

The temperature on the **CM's surface climbed up to 5,000 degrees Fahrenhei**t, but the heat shields protected the inner structure of the CM. The heat shield was ablative, which means that it was designed to melt and erode away from the CM as it heated up. From the ground, it would look as if the CM had caught on fire during its descent. In reality, the ablative covering is what kept the astronauts inside the CM safe -- the material diverted heat away as it vaporized.

Calculate Terminal Velocity of Command Module Prior to Parachute Deployment, v_{term}

Mass of the command module, CM: 5809kg
Drag coefficient, C_d, ~ 1.3 (See Apollo C_d Graph from following page). We time average this to get C_{d_avg}
Area of the CM's heat shield: 11.631 m2 (125.2 ft2)
Altitude Drogue parachutes deployed: 24,000 feet (7315.2m).
Density of the air at that altitude: 0.57 kg/m3.

$m_{CM} := 14690\text{kg} \qquad C_{d\_avg} := 1.3 \qquad A_{CM} := 11.6\text{m}^2 \qquad \rho_{air\_chute} := 0.57\frac{\text{kg}}{\text{m}^3}$

$v_{term} := \sqrt{\frac{2\cdot m_{CM}\cdot g}{\rho_{air\_chute}\cdot A_{CM}\cdot C_{d\_avg}}} \qquad v_{term} = 183.083\frac{\text{m}}{\text{s}} \qquad v_{term} = 409.545\cdot\text{mph}$

Energy Dissipated by Heat Shield to Atmosphere during Re-Entry

$\Delta KE(v_i, v_f) := \frac{1}{2}m_{CM}\cdot v_i^2 - \frac{1}{2}m_{CM}\cdot v_f^2 \qquad Heat\_Loss := \Delta KE\left(4.174\cdot\frac{\text{km}}{\text{s}}, 114\cdot\frac{\text{m}}{\text{s}}\right)$

$Heat\_Loss = 1.212\times10^8\cdot\text{BTU} \qquad Heating\_Load := \frac{Heat\_Loss}{A_{CM}} \qquad Heating\_Load = 9.707\times10^5\cdot\frac{\text{BTU}}{\text{ft}^2}$

$Heat\_Loss = 1.279\times10^5\cdot\text{Mega}\cdot\text{J}$

This is a tremendous amount of heat that must be dissipated by the vehicle. What If the vehicle were made of solid aluminum? Al has a heat capacity of 921 J/kg-K or 0.22 BTU/lb-°F. The rise in temperature would be:

$\Delta T_{rise} := \frac{Heat\_Loss}{m_{CM}\cdot 0.22\frac{\text{BTU}}{\text{lb}\cdot\text{F}}} \qquad \Delta T_{rise} = 1.701\times10^4\,\text{F}$

The boiling point of Al is just 4,478 F. If we did not have some way of dissipating this heat, the vehicle would melt, then and boil away.

Apollo Ablative Shield Temperature Rise

$m_{shield} := 700\text{lb} \qquad \Delta T_{rise} := \frac{Heat\_Loss}{m_{shield}\cdot\left(8.8\cdot10^6\frac{\text{J}}{\text{kg}\cdot\text{K}}\right)} \qquad \Delta T_{rise} = 45.764\,\text{K}$

The atmosphere acted like a braking system on the spacecraft. To further slow the CM's descent, the spacecraft used mortar-deployed parachutes.

Parachute deployment occurred at 195 hours, 13 minutes.

The pilot chutes are deployed at about 10,000 feet (3.05 km) by a barometric switch, pulling the three main parachutes from their containers. The ELS was designed so the drogue chutes slow the descent down to roughly 200 km/h (124 mi/h) before the pilot chutes pull the main chutes, eventually slowing down the CM to 22 miles per hour (35 km/h) for splashdown and to roughly 24.5 mi/h
(**39.5 km/h**) with only two main chutes properly deployed, as it happened during the Apollo 15 splashdown.
After a flight of 195 hours, 18 minutes, 35 seconds - about 36 minutes longer than planned - Apollo 11 splashed down in the Pacific Ocean, 13 miles from the recovery ship USS Hornet.

During a water impact the CM deceleration force will vary from 12 to 40 G's. of the waves and the CM's rate of descent. A major portion of the energy (75 to 90 percent) is absorbed by the water and by deformation of the CM structure.
The module's impact attenuation system reduces the forces acting on the crew to a tolerable level. The impact attenuation system is part internal and part external. The external part consists of four crushable ribs (each about 4 inches thick and a foot in length) installed in the aft compartment.

XXII. Simulate Drag Force or Drag Coefficient on CM - Five Different Ways

We call the forward part of a space vehicle that pushes the air out of the way the nose cone. The air resistance for the nose cone is determined by a factor called the Drag Coefficient. The Drag Force is given by the relation:

Drag Force or Drag = $C_d \rho v^2 A/2$

where Cd is the drag coefficient, v is the velocity, ρ is the density of air, and A is the frontal area. The drag also depends on the angle that the nose cone makes with the vertical. It can be measured from wind tunnel testing or data from the firing of projectiles. It is considerably more difficult for the case of rocket re-entry where the density of air varies with altitude and heating causes blow torch level air temperatures.

Note from the above equation the knowing only the Drag Coefficient is not sufficient to obtain the Drag Force. We must also know how the air density. But in the case of a rocket, the air density changes significantly with altitude. Thus we also need to know how the altitude changes.But the altitudes changes with both time and velocity. The density also changes with air boundary layer temperature, which is a function of heatflow to CM.

We will get values of the Drag Force by 5 different methods:

First we must model the variation of air density with respect to altitude

0. Atmospheric Variation of Altitude (z) of Temp(z), Pres(z), and ρ(z)
1. Variation of Cd with Mach Number from tables. Mach Number is the velocity in multiples of speed of sound
   From this we can create a function to approximate the variation of Cd with Mach Number.
2. Reentry Data: Velocity, Altitude, vs time Acceleration is Δv/Δt or slope. Data: *Apollo 8 Mission Report*
3. Theoretical Calculation of maximum acceleration. This does not model the altitude.
4. Constant Cd with variation of density ρ with altitude Obtain altitude from velocity or time data from method 2
5. Constant Cd with constant value of density ρ

The method that is ultimately successful is #2, that is,determine the acceleration and then simulate it.

0. Atmospheric Variation with Altitude (z) of Temp(z), Pres(z), and ρ(z)

Atmosphere model: US 1976 Standard Atmosphere (z < 11,000 m): Troposphere

Temperature (K) -> $T(z) := 288.149 - .00649 \cdot \frac{z}{m}$

Pressure (Pa) -> $p(z) := 101325 \cdot \left(1 - 2.26 \cdot 10^{-5} \cdot \frac{z}{m}\right)^{5.25}$

Density (kg/m³) -> $\rho(z) := \frac{p(z \cdot 10^3)}{T(z \cdot 10^3) \cdot 287} \cdot \frac{kg}{m^3}$

The ideal gas law in molar form, which relates pressure, density, and temperature:

$$P = \rho R_{\text{specific}} T$$

Speed of Sound (m/sec) $v_s(z) := \sqrt{401.8 \cdot T(z)}$

$\rho(1ft) = 1.19 \frac{kg}{m^3}$

1. Variation of Drag Coefficient with Velocity (Mach 0 to 10):

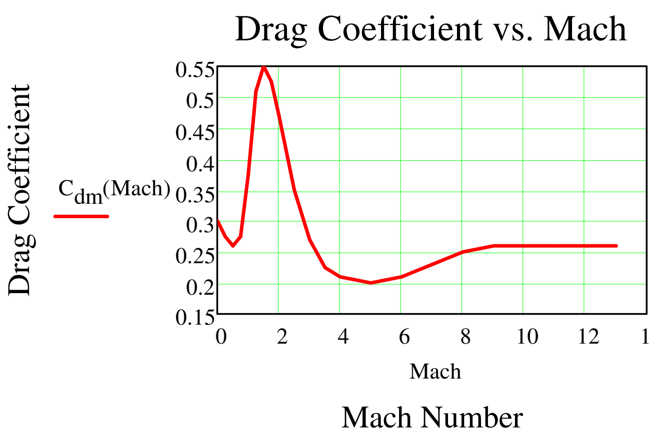
$Mach1 := 340 \frac{m}{s}$

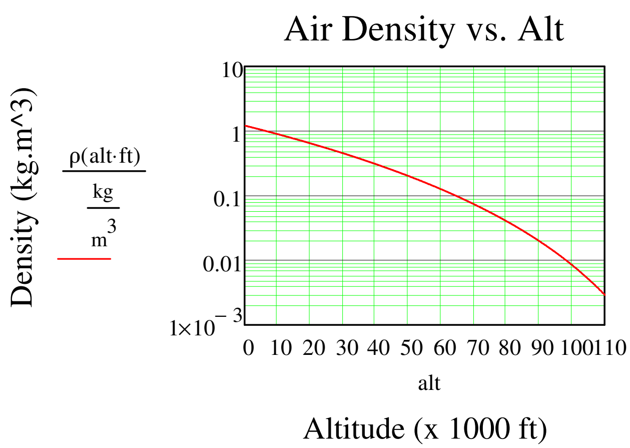
$Mach1 = 760.558 \cdot mph$

$Mach := (0\ \ 0.25\ \ 0.5\ \ 0.75\ \ 1\ \ 1.25\ \ 1.5\ \ 1.75\ \ 2\ \ 2.5\ \ 3\ \ 3.5\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12\ \ 13)^T$

$C_d := (0.3\ \ 0.275\ \ .26\ \ .275\ \ .375\ \ .51\ \ .55\ \ .525\ \ .47\ \ .35\ \ .27\ \ .225\ \ .21\ \ .2\ \ .21\ \ .23\ \ .25\ \ .26\ \ .26\ \ .26\ \ .26\ \ .26)^T$

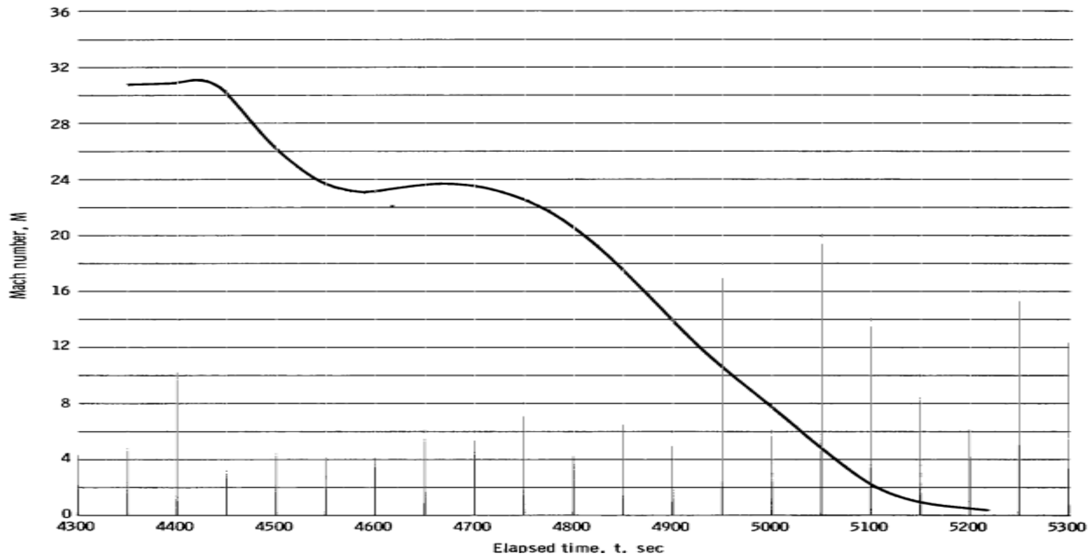
Interpolate for Altitude vs Mach # -> $C_{dm}(M) := interp(cspline(Mach, C_d), Mach, C_d, M)$ $\quad C_{dm}(13) = 0.26$



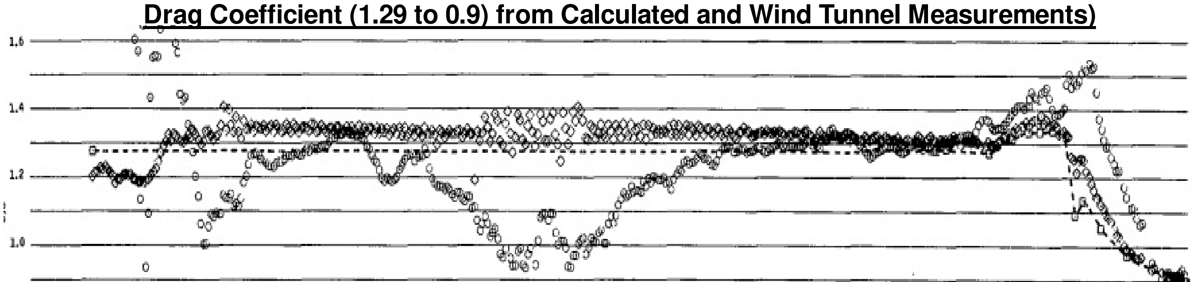


XXIII. Apollo Re- Entry: Velocity, Altitude, CD Flight Data vs. time

Apollo Time History of Trajectory Parameters AS-202

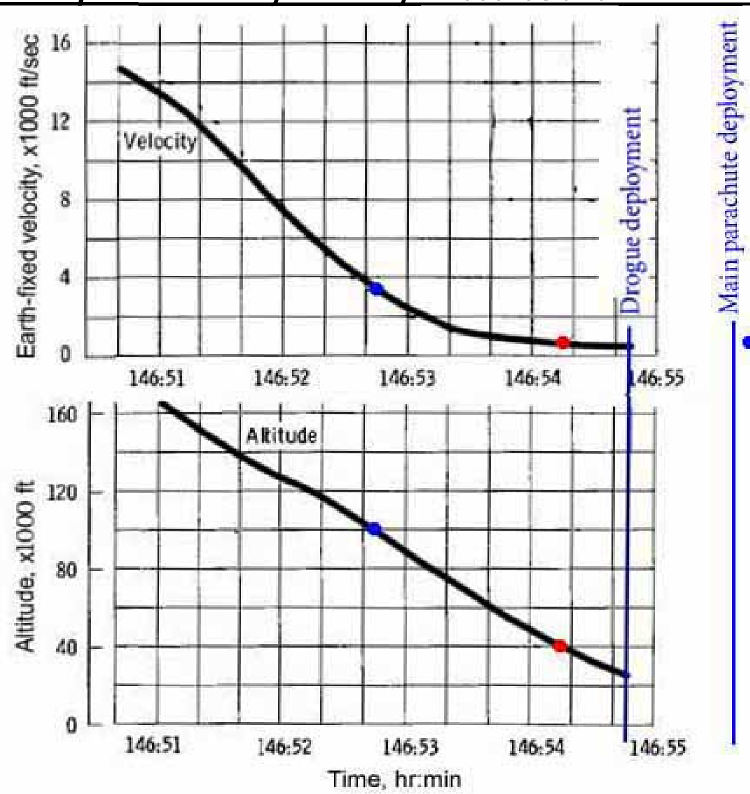


Drag Coefficient (1.29 to 0.9) from Calculated and Wind Tunnel Measurements)



Elapsed time from 4300 to 5220 seconds

***2. Apollo 8 Mission Report*: Re-entry Velocity x 1000 ft/s and Altitude x 1000 ft vs. time**



2. Apollo 8 Mission Report, pg 5-22: Re-entry Altitude x 1000 ft vs Velocity x 1000 ft/s

Put Data in Functional Form: **Alt(v) & Vel(t)**. Mach Number Velocity Decreased from 31 to 0

$t_{dn} := (0\ \ 20\ \ 39\ \ 60\ \ 80\ \ 100\ \ 118\ \ 128\ \ 130\ \ 150\ \ 160\ \ 180\ \ 190\ \ 200\ \ 220\ \ 240)^T$

Smooth Vel & Alt Data

$vel_{dn} := (14.8\ \ 13.5\ \ 12\ \ 10\ \ 8\ \ 5.8\ \ 4\ \ 3.5\ \ 3\ \ 2.8\ \ 1.8\ \ 1.7\ \ 1\ \ .8\ \ 0.79\ \ 0.7)^T$ $\quad vel_{dns} := \mathrm{ksmooth}(t_{dn}, vel_{dn}, 40)$

$alt_{dn} := (175\ \ 160\ \ 158\ \ 140\ \ 130\ \ 120\ \ 105\ \ 100\ \ 97\ \ 90\ \ 78\ \ 60\ \ 58\ \ 50\ \ 38\ \ 28)^T$ $\quad alt_{dns} := \mathrm{ksmooth}(t_{dn}, alt_{dn}, 40)$

Reverse Order, Make data ascending for Interp Fn $\quad n := 0, 1 .. 15 \quad veldni_n := vel_{dn_{15-n}} \quad altdni_n := alt_{dn_{15-n}}$

Interpolate Data for Altitude vs Velocity: $\quad Alt_{dni}(V) := \mathrm{interp}(\mathrm{cspline}(vel_{dni}, alt_{dni}), vel_{dni}, alt_{dni}, V)$

Add Units of fps $\quad fps := ft \cdot sec^{-1} \quad Alt_{dn}(v) := Alt_{dni}\left(\dfrac{v}{10^3 fps}\right) 10^3 ft \quad Alt_{dn}(10^3 fps) = 1.768 \times 10^4\ m$

2. Break Velocity Data, vel, into 2 Regions of different Slopes or Accels - Below Graph Shows 2 Slopes

This Mathcad "slope" function finds the slope

$$accel1 := \mathrm{slope}(\mathrm{submatrix}(t_{dn}, 0, 6, 0, 0), \mathrm{submatrix}(vel_{dn}, 0, 6, 0, 0)) \cdot \frac{1000ft}{s^2}$$

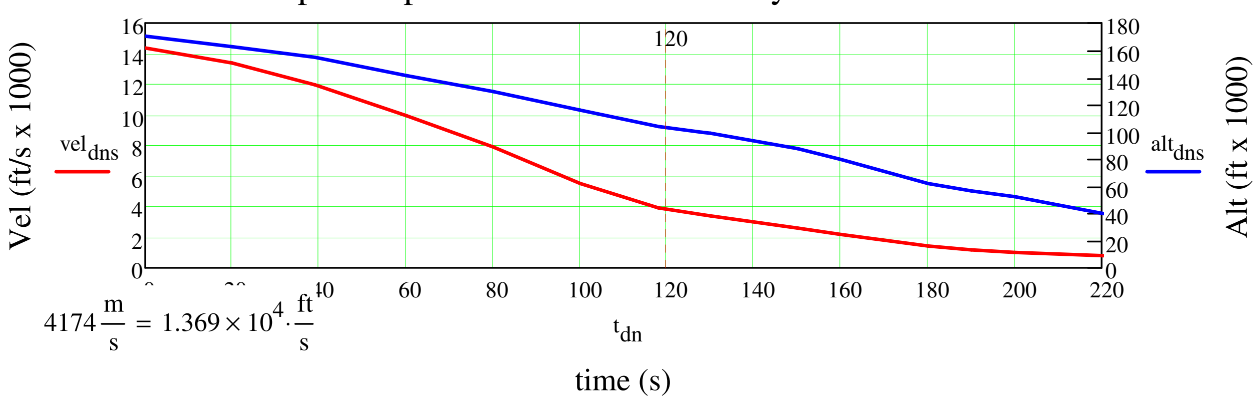
$$accel2 := \mathrm{slope}(\mathrm{submatrix}(t_{dn}, 7, 13, 0, 0), \mathrm{submatrix}(vel_{dns}, 7, 13, 0, 0)) \cdot \frac{1000ft}{s^2}$$

Acceleration1 for 0 to 120s $\quad accel1 = -2.9 \cdot g$

Acceleration2 for 120 to 200s $\quad accel2 = -1.11 \cdot g$

See that Blue Velocity Curve has two slopes = Δv/Δt. Interpret this as accel1 (-2.9g) and accel2 (-1.11g)

Plot of Apollo Splashdown Data: Velocity & Alititude vs.time



$4174 \frac{m}{s} = 1.369 \times 10^4 \cdot \frac{ft}{s}$

3. Theoretical Maximum De-Acceleration from Atmospheric Drag

Parameters for the Command Module

Earth Atmospheric Scale Altitude β - Characterizes Density Profile

Ballistic Coefficient, BC: C_d determines complex shape dependencies.

$A_{cm} := 11.5m^2$

$BC := \dfrac{m_{CM}}{C_d \cdot A_{cm}}$

Drag $= C_d\, \rho\, v^2 A/2$

$\rho_{air\_avg} := 0.5 \dfrac{kg}{m^3}$

$\rho 0_{air} := 1.225 \dfrac{kg}{m^3}$ $\quad C_d := 1.29 \quad \gamma := 50deg$

$BC = 990.226 \dfrac{kg}{m^2}$ $\quad \beta := \dfrac{0.000139}{m}$

Maximum Acceleration from Atmospheric Drag, a_{max}: γ is the vehicles's flight-path angle

$$a_{max}(v_{re\_entry}, \gamma) := \frac{v_{re\_entry}^2 \cdot \beta \cdot \sin(\gamma)}{2e}$$

$$alitiude_{amax}(\gamma) := \frac{1}{\beta} \cdot \ln\left(\frac{\rho_{air\_avg}}{BC \cdot \beta \cdot \sin(\gamma)}\right)$$

Reference: FAA Medical Studies:4.1.7
Returning from Space - Re-entry
Hi Speed Drag Calculations

3. This gives a maximum acceleration of 3.4 g. This value is close to accel from method #2.

$a_{max}\left(3.9 \dfrac{km}{s}, 5deg\right) = 3.456 \cdot g$

XXV. Atmospheric Braking & Splashdown CM Acceleration and Velocity

Simulate Re-Entry Velocity Profile for CM

$v_{entry} := 4174\frac{m}{s}$ $\quad v_{entry} = 9.337\times 10^{3}\cdot mph$

Spacecraft Geometry
$Diam := 12ft + 10in$
$m_{CM} := 14690kg$

Area Shield, As
$A_s := \pi\cdot\left(\frac{Diam}{2}\right)^2$

$D = C_d\ v^2\ A/2$ $\quad mps := \frac{m}{s}$ $\quad kmps := 10^3\frac{m}{s}$

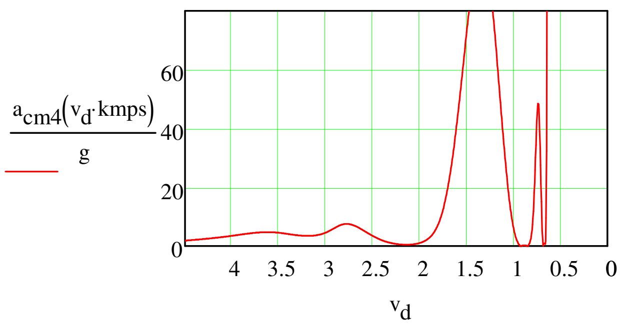
Simulate Drag Coefficient as a Function of how Altitude - Density varies with velocity to get a(v)

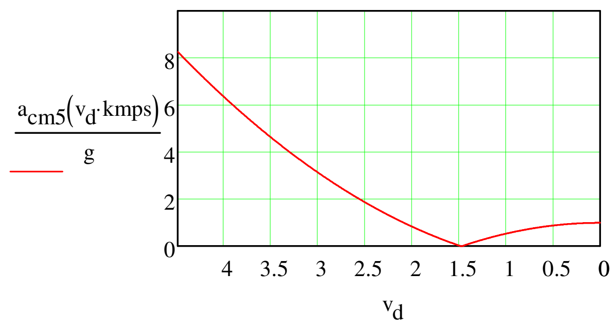
4. Drag Variable Density(Alt): $Drag(v) := \frac{1}{2}\cdot 1.29\cdot\rho\left(Alt_{dn}\left(\frac{v}{3.3}\right)\cdot 10^{-3}\right)\cdot(v)^2\cdot A_s$ $\quad a_{cm4}(v) := \left|g - \frac{Drag(v)}{m_{CM}}\right|$

5. Drag Fixed Altitude @ 100,000 ft : $DRAG(v) := \frac{1}{2}\cdot 1.29\cdot\rho(100ft)\cdot v^2\cdot A_s$ $\quad a_{cm5}(v) := \left|g - \frac{DRAG(v)}{m_{CM}}\right|$

D(v): Drag Coeff from **Density as Fn of Altitude** acm4 is not very stable. root function does not converge D(v).

5. Constant: acm5 is stable, but max g ~22 root function does not converge





5. Constant Drag Coefficient Velocity vs time

$V := 0\cdot mps$ $\quad vel_{cm5}(t) := root\left(t\cdot sec - \int_{4000}^{V}\frac{mps}{-a_{cm5}(V\cdot mps)}\,dV, V\right)\cdot mps$

$vel_{cm5}(200) = \blacksquare$

Convergence tol = 0.1

2. Simulate Splashdown Drag Data
Accel. asim, varied from 2.9 to 1.1g

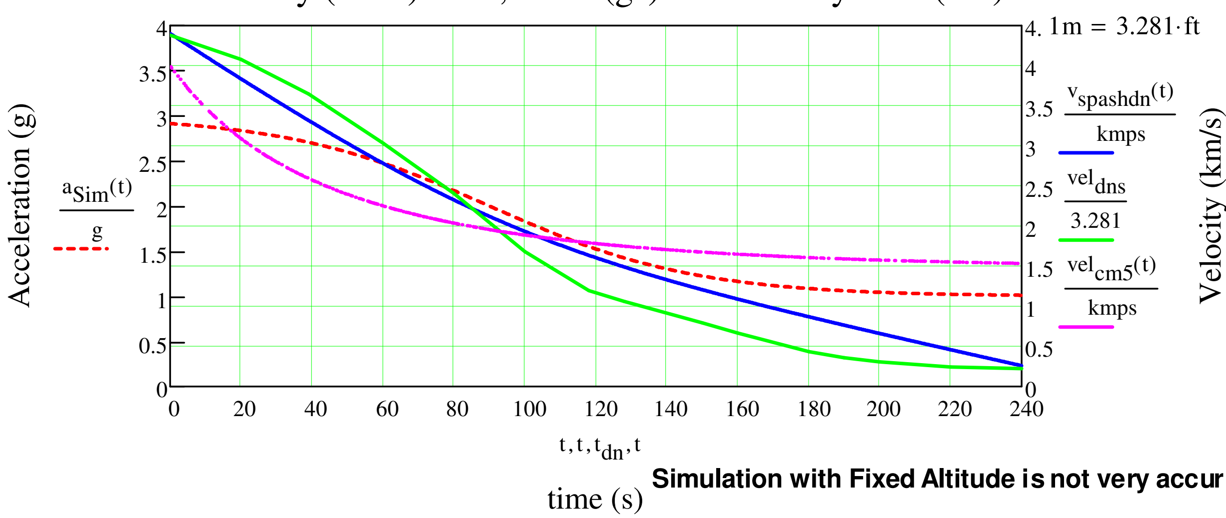
$a_{Sim}(t) := \left[\frac{2}{1 + e^{-\left(\frac{90-t}{29}\right)}} + 1\right]\cdot g$

Simulation of CM Velocity from Simulation of Apollo 8 Splashdown Drag Data

$v_{spashdn}(t) := 4400\frac{m}{s} - \int_0^t a_{Sim}(t)\,dt\cdot s$ $\quad v_{spashdn}(119.9) = 1.607\cdot\frac{km}{s}$ $\quad 14.8\cdot 1000\frac{ft}{s} = 4.511\cdot\frac{km}{s}$

#2. Simulation of Apollo CM Drag Acceleration asim and Resultant Velocity

Velocity (km/s) Blue, Accel(gs) vs. Re-entry time (sec)



Simulation with Fixed Altitude is not very accurate.

XXV. Trajectory Sim of Command Module (CM) from Moon to Earth Splashdown

XV Simulation of Apollo Command Module (CM) Trajectory from Moon to Earth Splashdown

This Simulation Uses the Differential Equation Solving Methodology detailed in:

arXiv:1504.07964 "***Motion of the planets: the calculation and visualization in Mathcad***", Valery Ochkov, Katarina Pisačić

$kg := 1 \quad m := 1 \quad s := 1 \quad N := 1 \quad s := 1 \quad min := 60s \quad hr := 3600s \quad km := 1000m \quad kmps := km \quad kph := \frac{km}{hr}$

Length of Simulation

$deg := \frac{2\pi}{360} = 0.017 \quad FRAME := 999 \quad t_{end} := \frac{185min}{999} \cdot FRAME + 1 \quad \mathbf{t} := t_{end} = 3.0836 \cdot hr \quad mph := 0.447 \cdot 10^{-3} kmps$

$v_0 := 3.8km \quad G := 6.67384 \cdot 10^{-11} \cdot \frac{N \cdot m^2}{kg} \quad R_e := 6370km \quad CM_0 := 13600kg \quad kgf := 9.80665N$

Atmospheric Drag, D: Time and Force

$t_1 := 179min \quad \Delta t := 5min \quad t_2 := t_1 + \Delta t \quad t_{drag} := t_1, t_1 + \frac{t_2 - t_1}{700} .. t_2 \quad D_r := CM_0 \cdot 2.5 \cdot kgf \quad D(t) := if(t_1 < t < t_2, D_r, 0kgf)$

Define Gravitational and Dynamics Equations for Earth and CM

$$\begin{pmatrix} m_e & x_{0e} & y_{0e} & x'_{0e} & y'_{0e} \\ m & x_0 & y_0 & x'_0 & y'_0 \end{pmatrix} := \begin{pmatrix} 5.972 \cdot 10^{24}\ kg & 0\ m & 0\ m & 0\ kph & 0\ kph \\ 13600kg & -50000\ km & R_e + 12439km & v_0 & 0kph \end{pmatrix}$$

$t_2 = 184 \cdot min \quad t_{end} = 1.11 \times 10^4 \quad CM(t) := CM_0 \quad 95 \cdot 60 = 5.7 \times 10^3$

Given

$x(0s) = x_0 \quad v_x(0s) = x'_0$

$y(0s) = y_0 \quad v_y(0s) = y'_0$

Set of Differential Guidance Equations for CM

$$CM(t) \cdot \left(\frac{d}{dt} v_x(t)\right) = -G \cdot \frac{CM(t) \cdot m_e \cdot x(t)}{\left(\sqrt{x(t)^2 + y(t)^2}\right)^3} - \frac{D(t) \cdot y(t)}{\sqrt{x(t)^2 + y(t)^2}} \qquad v_x(t) = x'(t)$$

$$CM(t) \cdot \left(\frac{d}{dt} v_y(t)\right) = -G \cdot \frac{CM(t) \cdot m_e \cdot y(t)}{\left(\sqrt{x(t)^2 + y(t)^2}\right)^3} + \frac{D(t) \cdot x(t)}{\sqrt{x(t)^2 + y(t)^2}} \qquad v_y(t) = y'(t)$$

Differential Equation Solver

$$\begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix} := Odesolve\left[\begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix}, t, t_{end}\right]$$

$t := 0s, \frac{t_{end}}{7000} .. t_{end} \quad t_{spdn} := 0s, \frac{t_2}{7000} .. t_2 \quad \alpha := 0, \frac{2\pi}{1000} .. 2\pi$

$t_{orb} := t_2, t_2 + \frac{t_2 - t_1}{70} .. t_{end} \quad \mathbf{D} := D(t_{end}) \quad R_{LM}(t) := \sqrt{x(t)^2 + y(t)^2}$

$D_x(t) := \frac{D(t) \cdot y(t)}{\sqrt{x(t)^2 + y(t)^2}} \quad D_y(t) := \frac{D(t) \cdot x(t)}{\sqrt{x(t)^2 + y(t)^2}} \quad m_s := CM(t_{end})$

$Y_{dot} := \left(y(t_2 + 40s) \quad y(t_{end})\right)^T$

$F_x := -D_x(t_1 + 10s) \quad F_y := -D_y(t_1 + 10s) \quad F_y = -2.901 \times 10^5 \cdot kg \quad \mathbf{D} = 0 \cdot kgf \quad F_x = -16758.5 \cdot kgf \quad X_{dot} := \left(x(t_2 + 40s) \quad x(t_{end})\right)^T$

Simulation of Apollo Command Module Trajectory from Moon to Earth Splashdown

Apollo 11 entered the atmosphere at a
height of 121.9 km & speed (s) 11.1 km/s. Sim is a good match.

Results of Simulation for Re-Entry Height and Speed
See Red Band in Below Plot for Atmospheric Entry
where Atmospheric Drag Forces and Parachute Deploys

$h(t) := R_{LM}(t) - R_e$

$R_{LM}(t_1) = 6494 \cdot km$ $\quad h(t_1) = 124.3 \cdot km$

$R_{LM}(t_2) = 6283 \cdot km$ $\quad s(t_1) = 11.011 \cdot kmps$ $\quad s(t_2) = 4.113 \cdot kmps$

$\frac{t_1}{min} = 179$

$s(t) := \sqrt{v_x(t)^2 + v_y(t)^2}$ $\quad v_{escape} = 11.182\ kmps$

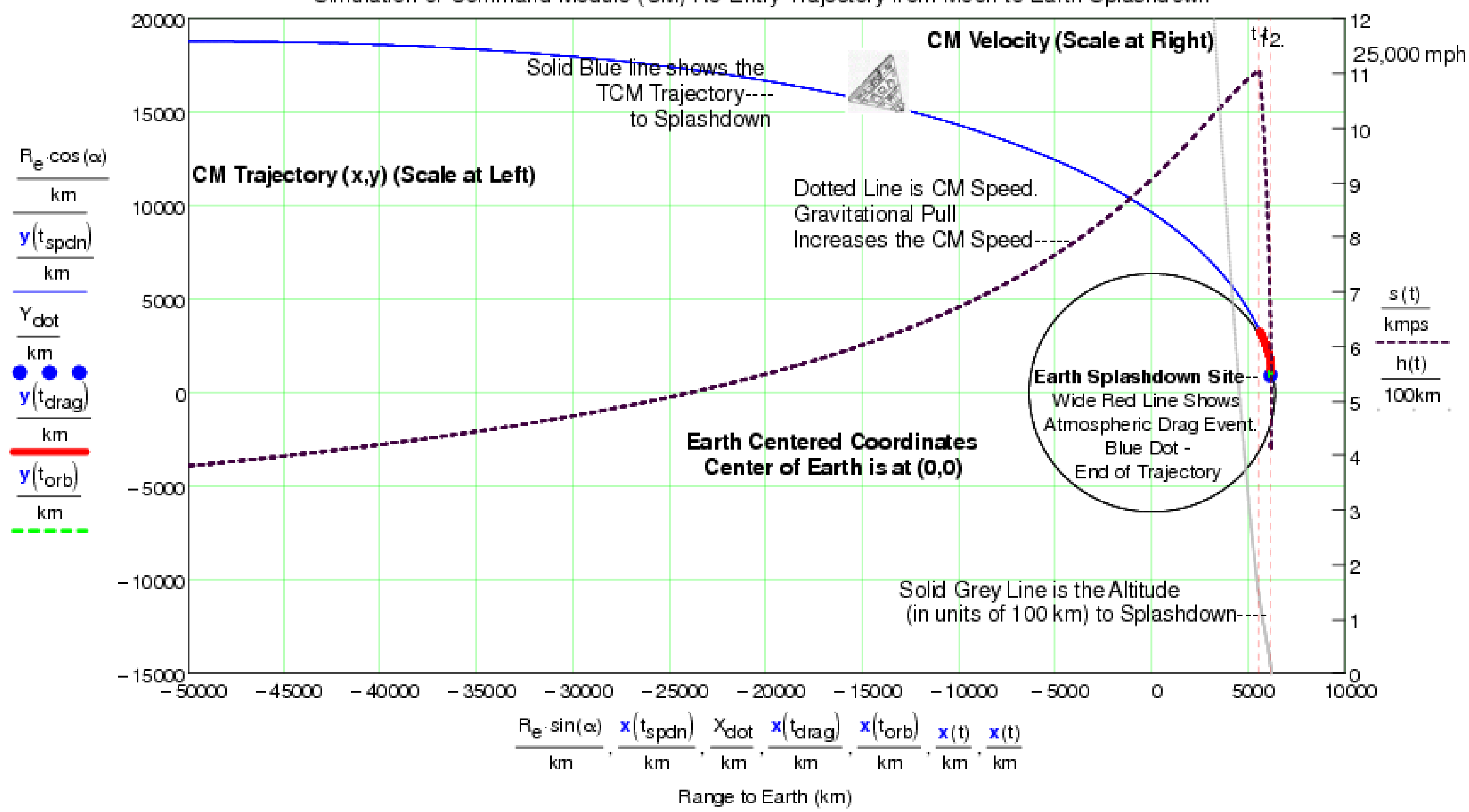
$s(t_2) = 4.113 \cdot kmps$ $\quad s(t_{end}) = 4.281 \cdot kmps$

$s(t_2) = 9.2 \times 10^3 \cdot mph$ $\quad s(t_{end}) = 9.578 \times 10^3 \cdot mph$

$x(t_1) = 5.596 \times 10^3 \cdot km$ $\quad x(t_2) = 6.179 \times 10^3 \cdot km$

Note: The Earth's Atmosphere is only about 100 km thick. The unit squares below are 5000 km.

Simulation of Command Module (CM) Re-Entry Trajectory from Moon to Earth Splashdown



XXVI. CM Splashdown: Parachute Terminal Velocity

Parachutes were a vital part of the Apollo Mission. They are deployed prior to splashdown for the Apollo command module, CM. The CM could not survive (remain intact) from a hard (high speed) crash into the water, unless slowed down to about 25 mph. For safety and redundancy, the CM had three main parachutes. Terminal Velocity occurs at the velocity where the Drag Force of the parachute equals the force of gravity, or the weight of the CM. Three main parachutes were huge. The amount of material in them was about 1/2 an acre.
The main parachutes were opened by the pilot parachutes at 10,000 feet, and slowed the rate of descent of the command module from 175 miles per hour to 22 miles per hour. In the case of the failure of one parachute (as happened to Apollo 15), the remaining two would be able to decelerate the module to 25 miles per hour. These parachutes held the command module at 27.5 degrees so the module's slanted corner would penetrate the water first, lessening the impact force. After splashdown, the risers of the main parachutes were cut and the parachutes released.

$$A_{chute} := \frac{1}{2}acre$$

$$D_{chute}(v, alt) := \frac{1}{2} \cdot C_d \cdot \rho(alt) \cdot v^2 \cdot A_{chute}$$

Terminal velocity at 10,000 ft occurs when:

$$D_{chute} = Weight$$

$$v_{terminal} := \sqrt{\frac{2m_{CM} \cdot g}{0.5 \cdot \rho(10ft) \cdot \frac{1}{2}acre}} \qquad v_{terminal} = 39.694 \cdot mph$$

At launch the Apollo spacecraft was 363 feet tall and weighed 6.2 million pounds. At splashdown it (the remaining command module) weighs 11,000 pounds and is 36 feet high. Just 0.2% of its original weight and 1/10th the height.

AstroDynamics Glossary and Keplerian Model

In the Keplerian model or elements, named after Johannes Kepler (1571-1630). satellites orbit in an ellipse of constant shape and orientation. The model relates **quantities measured from the earth** to **properties of an ellipse**. The ***key*** to this is that area is swept out at a constant rate in Kepler's model. That is, **(t-T)/A = Period/(π a b),** where T is the time of periapsis passage and A is the area swept out. The Earth is at one focus of the ellipse, not the center (unless the orbit ellipse is actually a perfect circle). The real world is slightly more complex than the Keplerian model, these are compensated by introducing minor corrections or perturbations. the six independent constants defining an orbit that were These constants are:

argument of perigee, ω -- angle from ascending nodes to perigee point along orbit, measured in direction of satellite's motion

eccentricity, e -- defines shape of orbit

inclination angle, i -- gives angle between a satellite's orbital plane and the equator

right ascension of the ascending node -- gives the rotation of orbit plane from reference axis

semi-major axis, a -- defines the max size of orbit

true anomaly, ν -- or θ defines satellite location on orbit.

Equation of Ellipse. Position r for angle θ **relative to the focus:** $$r(\theta) = \frac{a(1-e^2)}{1 \pm e\cos\theta}$$

Kepler's Equation: relates an orbit's Mean Anomaly (M) with its Eccentric Anomaly (E):
M(t) = E(t) - e sin (E(t)). Kepler's Equation is transcendental and therefore cannot be solved analytically for E(t). Instead, computers can be used to find the best value of E(t) that satisfies the equation for the known values of M(t) and e. Once E(t) is determined, the True Anomaly, n(t), can be determined. For example: **To compute the position of a point moving in a Keplerian orbit.**
Given: Observation gives that the body passes at (x,y) coordinates x = a(1 − e), y = 0, at time t = t0, then to find out the position of the body at any time, you first calculate the mean anomaly M from the time and the mean motion n by the formula $M = n(t - t_0)$, then compute the value E. This then gives the coordinates: x = a (cos(E) -e) and y = b sin(E)

Angular Momentum of Ellipse, h: Related to the semi-latus rectum, p. **p = h2/μ , where μ = G Me**

Apogee, $r_a = a(1 + e)$
Perigee (Periapsis), $r_p = a(1 - e)$ **,** Point **along the longest axis, a,** (but nearest to the body foci)

Ascending Node (): The precise point in a satellite's orbit that intersects the equatorial plane of the Earth as the satellite moves from the southern to the northern hemisphere (ascending).

Descending Node (☋): The precise point in a satellite's orbit that intersects the equatorial plane of the Earth as the satellite moves from the northern to the southern hemisphere (descending).

Argument of perigee: angle from ascending nodes to perigee point along orbit, measured in direction of satellite's motion. **M(t) = E(t) - e sin (E(t)).**

Astronomical Unit (AU): about 149,599,000 kilometers; the distance from the Earth to the Sun.

azimuth: angular distance in degrees measured in a clockwise direction from true north.

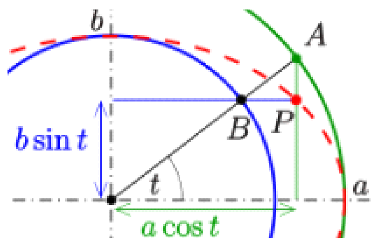
Ephemeris: an arrangement of a series of data points defining both the position and motion of a satellite

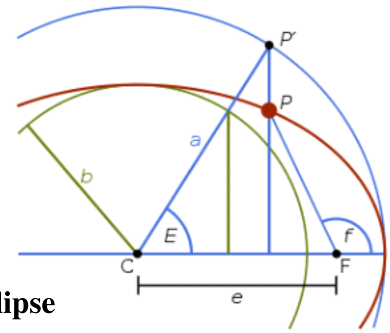
Eccentricity (e): defines how oval the satellite's orbit is. It is mathematically defined as the ratio of the orbit's focus distance (c) to the orbit's semi-major axis **(a). e = c/a**

Eccentric Anomaly (E): The angle, measured since perigee, based on the hypothetical position P' on the **circular orbit** defined by a line perpendicular to the major axis that **passes through the true position P** of the satellite and **intersects with the circular orbit at P'**. The Mean Anomaly is directly related to the Eccentric Anomaly through Kepler's Equation. **M(t) = E(t) e sin (E(t)).**
Ellipse is given by the equation: (x/a)2 + (y/b)2 = 1. Then from the right triangle in Figure: **cos E = x/a. Also, for** (x, y) = (a cos t, b sin t) $0 \leq t < 2\pi$, the parameter t is called the **eccentric anomaly**.

Eccentricity, e: $e = \frac{c}{a}$

c is the distance from the center to either foci, F.
where a and b are defined as semi-major and semi-minor axes





Focus Distance (c): The distance from the **true center** of the orbit **ellipse** to the center of the **Earth**.

Mean Anomaly (M): M was defined by Kepler. It is the angle measured since perigee that would be swept out by the satellite **if** its orbit were **perfectly circular**. It was worked out by geometrical construction. If Pfa equals the Area (Pfa) swept out by the Ellipse from point a to P, then M = Pfa / (1/2 a*b). **This hypothetically constructed orbit** would assume the real orbit's semi-major axis and its period. The Mean Anomaly indicates where the satellite was in its orbit at a specific time.
The mean anomaly is convenient since it is a geometric quantity which is **directly proportional** to the **time**. **M = n * (t - τ),** where τ is the time at the perigee and n = 2π/T. **M(t) = E(t) e sin (E(t))** The **Mean Anomaly, M, equals the True Anomaly, ν,** ***for a circle***. By definition, **$M - M_0 = n^* (t - t_0)$**, where M_0 equals the True Anomaly of an ellipse, **ν**, at time 0, and n is the mean motion. It is the fraction of an orbit period that has elapsed since the perigee

Mean Motion (n): Number of orbits the satellite completes about the Earth in exactly 24 hours or where μ is equal to G * Mass. $n = \sqrt{\frac{\mu}{a^3}}$

Period (T): The time required for the satellite to orbit the Earth (or a planet) once.

Perturbations: small adjustments made to the Keplerian model of a satellite's orbit; due to Earth's gravity and drag, a satellite's orbit is not a perfect ellipse of constant shape and orientation.

Semi-Major Axis (a): The distance from the **center** of the orbit ellipse to satellite's apogee or perigee point. This is also defined as the average distance of the satellite from the Earth's center. It can be found from Kepler's 3rd Law: a^3 is proportional to T^3 for n = the Mean Motion of the satellite's orbit.
or by Newton's Laws: **$a^3 = GM / (2\pi n)^2$** or **$a^3 = G\ M\ T^2 / 4\pi^2$** = kT^2 where $k = GM / 4\ \pi^2$
The Semi-Major axis, **a** depends only on the Specific Total Energy (energy/mass), $\mathcal{E} = v_0^2 - \mu/r_0^2$ at a
Semi-Minor Axis (b): The shortest distance from the **true center** of the orbit ellipse to the orbit path.

Semi-Latus Rectum, p: **$p = a^*(1 - e^2)$** It is the y coordinate of the ellipse

Time of Flight, t T: Given the above, we can derive for the time of flight
$$t - T = (E - e\cdot\sin(E))\cdot\sqrt{\frac{a^3}{G\cdot M}}$$

True Anomaly (ν): The true angle, measured since the perigee that the satellite sweeps out while **orbiting the Earth**. See polar equation of an ellipse.