

# Collision Planar Impact Analysis

This collision model is based on the **Planar Impact Mechanics** Model (PIM)<sup>1</sup> of Raymond M. Brach. The PIM model is a time forward calculation that inputs linear and angular velocities of two vehicles colliding at a point C along a normal surface at angle  $\Gamma$ , given two constraints for the **Coefficient of Restitution,  $\epsilon$** , and the **Impulse Ratio,  $\mu$** . PIM then solves this for the final velocities.

The coefficient of restitution is the ratio of the Final Normal Relative Velocity at the common point of contact of the collision (C), normal to  $\Gamma$ ,  $V_{cn}$ , to the initial normal relative velocity,  $v_{cn}$ . Then the coefficient of restitution,  $\epsilon$ , equals  $-V_{cn}/v_{cn}$ . Equivalently,  $\epsilon = (V_{2n} - V_{1n})/(v_{2n} - v_{1n})$  or  $\epsilon = -Pr/Pi$ .

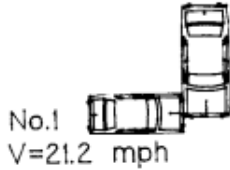
The Impulse ratio gives the **tangential common velocity condition**, that is, where  $V_{1Ct} - V_{2Ct} = 0$ . Let  $\beta$  be the fraction of  $\mu_0$  that is used for the reconstruction. Then  $\mu = \beta * \mu_0$ , where  $\beta$  is a measure of the intervehicle sliding.

## Goals of this Analysis:

- Develop a Mathcad implementation and an explicit and deeper investigation of the PIM.
- Express results as functions of  $\epsilon$  and  $\beta$ , for example, as  $V_{x1}(\epsilon, \beta)$ , for the final x-velocity of car 1.
- We can then calculate and show the sensitivities of final results to the coefficient of restitution  $\epsilon$ .
- Find the Optimum values of  $\epsilon$  and  $\mu$  to minimize the Least Squares error of the final velocities.
- Verify results by comparing against the RICSAC 9 collision and Fig 7.14<sup>1</sup>.
- This functional form allows the final results to be graphed as shown on pages 8 and 9.

## Validation: Analysis of RICSAC Collision #9 Brach<sup>1</sup> Fig 7.14

The Research Input for Computer Simulation of Automobile Collisions (RICSAC) was a federally funded research project by CALSPAN Corporation's Advanced Technology Center. The intent was to provide critical data for validating existing and future computer analysis and simulation programs. There were 12 series of staged collisions configurations. Collision #9 was an oblique 90 degree collision as illustrated below.

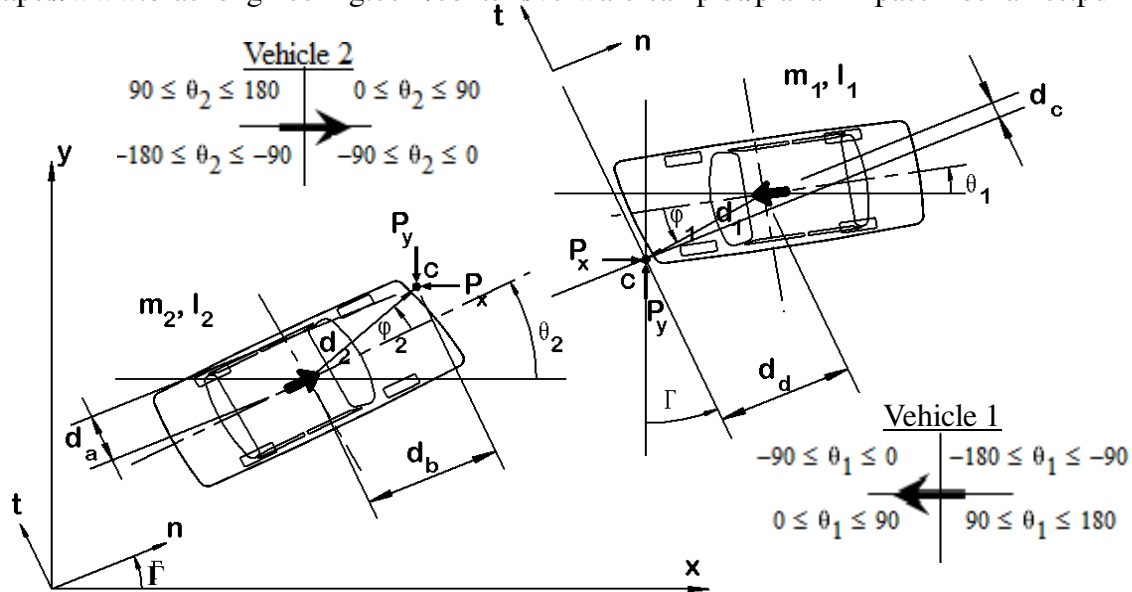
|                                                                                                                                                                                              |                                                        |         |                                                                                      |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------|---------|--------------------------------------------------------------------------------------|
| 9                                                                                                                                                                                            | No. 1 - '74 Honda Civic<br><br>No. 2 - '74 Ford Torino | Oblique |  |
| $\epsilon := 0.245$ $\mu := 0.714$ $e_m := -0.914$ $E_{loss\%} := 40$ $\Omega_{Honda} := -3.142 \cdot \frac{\text{rad}}{\text{s}}$ $\Omega_{Ford} := 7.86 \cdot \frac{\text{rad}}{\text{s}}$ |                                                        |         |                                                                                      |

Note: Brach notation<sup>1</sup> has initial  $v_{x1}$  as negative,  $\theta_1 = 0$  degrees,  $\theta_2 = 90$  degrees, and  $\Gamma = 0$  degrees. Detailed RICSAC 9 input conditions in the PIM model are given in the table on the following page. The orientation used in the Brach Fig 7.14 for RICSAC 9 differs from the above.

## PIM - Forward Calculation Conservation of Momentum Model

The drawing below shows the general relationships between the Brach variables

<https://www.brachengineering.com/content/vcrware-samples/planar-impact-mechanics.pdf>



Given:

1. Initial velocity components:  $v_{1x}$ ,  $v_{1y}$ ,  $v_{2x}$ ,  $v_{2y}$ ,  $\omega$ , where  $\omega$  is relative to x or street axis.
2. Vehicle physical properties: Weight<sub>1</sub> (mass<sub>1</sub>), Weight<sub>2</sub> (mass<sub>2</sub>), and Moments of Inertia:  $I_1$ ,  $I_2$
3. Heading orientation angles:  $\theta_1$ ,  $\theta_2$  (relative to x axes). All angles, except  $\phi$ , are relative to x axis
4. Heading angles for **head on collision** have a datum of 0 degrees CCW for Vehicle1 moving to left
5. Collision damage characteristics of **common point C**:  $d_1$ ,  $d_2$ ,  $\phi_1$ ,  $\phi_2$ , (ds to CM,  $\phi$ s relative to  $\theta$ s)

The contact surface ( $\sim$  stiffness) is the time and space average of the deformed contact surface.

C is the common/impact center point of the intersection of the impulse and contact surface.

Energy loss:  $\epsilon$  (coeff of normal restitution), The normal impulse ( $P=\Delta p$ ) can be divided into 2 parts  $P_a$  during approach and  $P_r$  during rebound, such that  $\epsilon = -P_r/P_a$ ,  $\epsilon$  of kinetic coeff restitution

Equivalently,  $\epsilon = (V_{2n}-V_{1n})/(v_{2n}-v_{1n})$ . Guess  $\mu$  (ratio of tangential to normal impulse) Max  $\mu_0$ .

A normal and tangential coordinate system is referenced wrt a common contact or crush surface.

$\Gamma$  is the angle of the normal to the collision plane relative to the x or street axis. N like x, is + to Right

Input conditions are given in lower case and Output parameters are upper case.

Find: Final velocities  $V_{x1}$ ,  $V_{x2}$ ,  $V_{y1}$ , and  $V_{y2}$  and angular momenta,  $\Omega_1$ ,  $\Omega_2$ .

### Initial Conditions:

### Vehicle 1

### Vehicle 2

Weight/Mass, w/m  
Moment of Inertia, I  
Common pt. moment arm, d  
Angle @CM of hdg  $\theta$  to P,  $\phi$   
CCW Impulse "heading" angle,  $\theta$   
Initial  $v_x$   
Initial  $v_y$   
Angular velocity,  $\omega$

$$\begin{pmatrix} m_1 \\ I_1 \\ d_1 \\ \phi_1 \\ \theta_1 \\ v_{x1} \\ v_{y1} \\ \omega_1 \end{pmatrix} := \begin{pmatrix} 70.11 \text{ lb} \cdot \text{s}^2 \cdot \text{ft}^{-1} \\ 976 \cdot \text{ft} \cdot \text{lb} \cdot \text{s}^2 \\ 4.8 \cdot \text{ft} \\ 6 \\ 0 \\ -31.09 \cdot \text{ft} \cdot \text{s}^{-1} \\ 0 \text{ mph} \\ 0 \cdot \text{deg} \cdot \text{s}^{-1} \end{pmatrix}$$

$$\begin{pmatrix} m_2 \\ I_2 \\ d_2 \\ \phi_2 \\ \theta_2 \\ v_{x2} \\ v_{y2} \\ \omega_2 \end{pmatrix} := \begin{pmatrix} 152.21 \text{ lb} \cdot \text{s}^2 \cdot \text{ft}^{-1} \\ 3953.3 \cdot \text{ft} \cdot \text{lb} \cdot \text{s}^2 \\ 5.6 \cdot \text{ft} \\ -29.7 \\ 90 \\ 0 \cdot \text{ft} \cdot \text{s}^{-1} \\ 31.09 \cdot \text{ft} \cdot \text{s}^{-1} \\ 0 \cdot \text{deg} \cdot \text{s}^{-1} \end{pmatrix}$$

Angle of normal to collision plane,  $\Gamma$

$$\Gamma := 0$$

RICSAC 9 is a 90 degree ( $\theta_2$ ) collision .

### Centroidal Yaw Radii of Gyration, k

$$m_1 := \frac{W_1}{g} \quad m_2 := \frac{W_2}{g} \quad k_1 := \sqrt{\frac{I_1}{m_1}} = 3.731 \text{ ft} \quad k_2 := \sqrt{\frac{I_2}{m_2}} = 5.097 \text{ ft} \quad k_1^2 = 13.923 \text{ ft}^2 \quad k_2^2 = 25.974 \text{ ft}^2$$

$$\text{Center of Mass } m_C := \frac{m_1 \cdot m_2}{m_1 + m_2} \quad v_1 := \sqrt{v_{x1}^2 + v_{y1}^2} = 31.09 \cdot \frac{\text{ft}}{\text{s}} \quad v_2 := \sqrt{v_{x2}^2 + v_{y2}^2} = 31.09 \cdot \frac{\text{ft}}{\text{s}}$$

### Define Sin and Cos functions to work with degrees and not radians

Degrees to radians:  $\sin(\theta) := \sin(\theta \cdot \text{deg})$

$\cos(\theta) := \cos(\theta \cdot \text{deg})$

"deg" converts  
radians to degrees

### From the above diagram

The **common point of collision**, C, is located relative to the C of G by distances  $d_1$ ,  $d_2$  and **angles**  $\phi_1$ ,  $\phi_2$ .  **$d_a$  to  $d_d$**  are the momentum arms for normal and tangential **Impulse P** ( $\Delta p$ ) components.

$$d_a := d_2 \cdot \sin(\theta_2 + \phi_2 - \Gamma) \quad d_b := d_2 \cdot \cos(\theta_2 + \phi_2 - \Gamma) \quad d_a = 4.864 \cdot \text{ft}$$

$$d_c := d_1 \cdot \sin(\theta_1 + \phi_1 - \Gamma) \quad d_d := d_1 \cdot \cos(\theta_1 + \phi_1 - \Gamma) \quad d_b = 2.775 \cdot \text{ft}$$

$$d_c = 0.502 \cdot \text{ft}$$

$$d_d = 4.774 \cdot \text{ft}$$

### Given: The Normal and Tangential Velocity

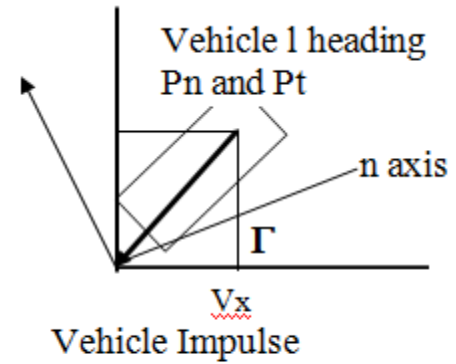
Components relative to the  $\Gamma$  normal axis

$$v_{n1} := v_{x1} \cdot \cos(\Gamma \cdot \text{deg}) - v_{y1} \cdot \sin(\Gamma \cdot \text{deg}) \quad v_{n1} = -31.09 \cdot \frac{\text{ft}}{\text{s}}$$

$$v_{t1} := v_{x1} \cdot \sin(\Gamma \cdot \text{deg}) + v_{y1} \cdot \cos(\Gamma \cdot \text{deg}) \quad v_{t1} = 0 \cdot \frac{\text{ft}}{\text{s}}$$

$$v_{n2} := v_{x2} \cdot \cos(\Gamma \cdot \text{deg}) + v_{y2} \cdot \sin(\Gamma \cdot \text{deg}) \quad v_{n2} = 0 \cdot \frac{\text{ft}}{\text{s}}$$

$$v_{t2} := v_{x2} \cdot \sin(\Gamma \cdot \text{deg}) + v_{y2} \cdot \cos(\Gamma \cdot \text{deg}) \quad v_{t2} = 31.09 \cdot \frac{\text{ft}}{\text{s}}$$



$$P_x = V_x - v_x, \quad P_y = V_y - v_y$$

$$P_n = P_x \cos(\Gamma) + P_y \sin(\Gamma)$$

### Velocity of "C" Common Point Normal Velocity, $v_{cn}$

Given: The Initial Common/Impact Velocity of Point of Collision point, C.

$$v_{cn1} := v_{n1} + d_c \cdot \omega_1 \cdot \text{deg} = -31.09 \cdot \frac{\text{ft}}{\text{s}} \quad v_{cn2} := v_{n2} - d_a \cdot \omega_2 \cdot \text{deg} = 0 \cdot \frac{\text{ft}}{\text{s}}$$

### Relative normal velocity vehicle 1 to vehicle 2

Eq. 6.46

$$v_{rn} := (v_{n2} - d_a \cdot \omega_2) - (v_{n1} - d_c \cdot \omega_1) = 31.09 \cdot \frac{\text{ft}}{\text{s}} \quad v_{cm} := v_{n1} + d_c \cdot R \cdot \omega_1 - v_{n2} + d_a \cdot R \cdot \omega_2 = -31.09 \cdot \frac{\text{ft}}{\text{s}}$$

$$v_{ct1} := v_{t1} - d_d \cdot \omega_1 \cdot \text{deg} = 0 \cdot \frac{\text{ft}}{\text{s}} \quad v_{ct2} := v_{t2} + d_b \cdot \omega_2 \cdot \text{deg} = 31.09 \cdot \frac{\text{ft}}{\text{s}}$$

Given: The Initial Common/Impact Velocity of Point of Collision, C.

$$v_{rn} := (v_{n2} - d_a \cdot \omega_2) - (v_{n1} - d_c \cdot \omega_1) = 31.09 \cdot \frac{\text{ft}}{\text{s}}$$

$$A := 1 + m_C \cdot \frac{d_c^2}{m_1 \cdot k_1^2} + m_C \cdot \frac{d_a^2}{m_2 \cdot k_2^2} = 1.3$$

$$B := m_C \cdot \frac{d_c \cdot d_d}{m_1 \cdot k_1^2} + m_C \cdot \frac{d_a \cdot d_b}{m_2 \cdot k_2^2} = 0.282$$

Eq. 6.46

Eq. 6.60  
to 6.62

$$C := m_C \cdot \frac{d_d^2}{m_1 \cdot k_1^2} + m_C \cdot \frac{d_b^2}{m_2 \cdot k_2^2} = 1.214$$

### Ratio of Initial Tangential to Normal Velocities:

$$r = \frac{v_{t2} - v_{t1}}{v_{n2} - v_{n1}} = 1$$

Angle between relative  
velocity and normal

$$r := \frac{v_{t2} - v_{t1}}{v_{n2} - v_{n1}} = 1$$

$$\frac{180}{\pi} \text{atan}(r) = 45$$

Eq. 6.60  
to 6.62

**See Pg 6 for Calculation of Optimum  $\varepsilon$** 

$$\varepsilon := 0.355 \quad \beta := 1$$

$$\text{Definition: } V_{2n} - V_{1n} = -\varepsilon (v_{2n} - v_{1n})$$

$$V_{crn} = -\varepsilon * v_{crn} \text{ (crn=point C, relative, normal)}$$

An important quantity is the **Impulse Ratio,  $\mu$** , that gives the **tangential common velocity condition**, that is, where  $V_{1Ct} - V_{2Ct} = 0$ . This gives the critical impulse ratio  $\mu_0$ .

Let  $\beta$  be the fraction of  $\mu_0$  that is used for the reconstruction. Then  $\mu = \beta * \mu_0$ , as shown below.

$$P_t = \mu P_n \quad \mu_{\max}(\varepsilon) := \frac{r \cdot A + (1 + \varepsilon) \cdot B}{(1 + \varepsilon) \cdot (1 + C) + r \cdot B} \quad \text{Eq. 6.60} \quad \mu_{\max}(\varepsilon) = 0.512 \quad \mu(\varepsilon, \beta) := \beta \cdot \mu_{\max}(\varepsilon)$$

$$d_e(\varepsilon, \beta) := d_c - \mu(\varepsilon, \beta) \cdot d_d \quad d_f(\varepsilon, \beta) := d_a - \mu(\varepsilon, \beta) \cdot d_b \quad \text{Eq. 6.59} \quad \mu(\varepsilon, 1) = 0.512$$

**q is the Brach rotation parameter**

$$q(\varepsilon, \beta) := (A - \mu(\varepsilon, \beta) \cdot B)^{-1} \quad \text{Eq. 6.55}$$

**Brach Planar Impact:** Given six initial conditions vx, vy,  $\omega$  for each vehicle plus  $\varepsilon$  and  $\mu$  Results in planar solution/equations for the 8 unknowns: Final Vx, Vy,  $\Omega$ , P) for each vehicle

$$\begin{aligned} \text{Eq. 6.48 to 6.53} \\ V_{n1}(\varepsilon, \beta) &:= v_{n1} + \frac{m_C \cdot (1 + \varepsilon) \cdot v_{rn} \cdot q(\varepsilon, \beta)}{m_1} & V_{t1}(\varepsilon, \beta) &:= v_{t1} + \frac{\mu(\varepsilon, \beta) \cdot m_C \cdot (1 + \varepsilon) \cdot v_{rn} \cdot q(\varepsilon, \beta)}{m_1} \\ V_{n2}(\varepsilon, \beta) &:= v_{n2} - \frac{m_C \cdot (1 + \varepsilon) \cdot v_{rn} \cdot q(\varepsilon, \beta)}{m_2} & V_{t2}(\varepsilon, \beta) &:= v_{t2} - \frac{\mu(\varepsilon, \beta) \cdot m_C \cdot (1 + \varepsilon) \cdot v_{rn} \cdot q(\varepsilon, \beta)}{m_2} \end{aligned}$$

$$V_1(\varepsilon, \beta) := \sqrt{V_{n1}(\varepsilon, \beta)^2 + V_{t1}(\varepsilon, \beta)^2} \quad V_2(\varepsilon, \beta) := \sqrt{V_{n2}(\varepsilon, \beta)^2 + V_{t2}(\varepsilon, \beta)^2}$$

$$V_1(\varepsilon, 1) = 9.669 \cdot \text{mph}$$

$$V_2(\varepsilon, 1) = 18.885 \cdot \text{mph}$$

$$\Omega_1(\varepsilon, \beta) := \omega_1 + m_C \cdot (1 + \varepsilon) \cdot v_{rn} \cdot d_e(\varepsilon, \beta) \cdot \frac{q(\varepsilon, \beta) \cdot 180}{\pi \cdot I_1}$$

Eq. 6.52 and 6.53

$$\Omega_1(\varepsilon, 1) = -199.702 \frac{1}{s}$$

$$\Omega_2(\varepsilon, \beta) := \omega_2 + m_C \cdot (1 + \varepsilon) \cdot v_{rn} \cdot d_f(\varepsilon, \beta) \cdot \frac{q(\varepsilon, \beta) \cdot 180}{\pi \cdot m_2 \cdot k_2^2}$$

$$\Omega_2(\varepsilon, 1) = 87.322 \frac{1}{s}$$

**Planar Impact (Rigid Body Impact) Mechanics****Impact  $\Delta V$  and Impulse**

$$\Delta V_1(\varepsilon, \beta) := \sqrt{(v_{n1} - V_{n1}(\varepsilon, \beta))^2 + (v_{t1} - V_{t1}(\varepsilon, \beta))^2} \quad \text{Impulse}_1(\varepsilon, \beta) := m_1 \cdot \Delta V_1(\varepsilon, \beta)$$

$$\Delta V_2(\varepsilon, \beta) := \sqrt{(v_{n2} - V_{n2}(\varepsilon, \beta))^2 + (v_{t2} - V_{t2}(\varepsilon, \beta))^2} \quad \text{Impulse}_2(\varepsilon, \beta) := m_2 \cdot \Delta V_2(\varepsilon, \beta)$$

$$P_{n1}(\varepsilon, \beta) := m_1 \cdot (V_{n1}(\varepsilon, \beta) - v_{n1}) \quad P_{t1}(\varepsilon, \beta) := m_1 \cdot (V_{t1}(\varepsilon, \beta) - v_{t1})$$

$$P_{n2}(\varepsilon, \beta) := m_2 \cdot (V_{n2}(\varepsilon, \beta) - v_{n2}) \quad P_{t2}(\varepsilon, \beta) := m_2 \cdot (V_{t2}(\varepsilon, \beta) - v_{t2})$$

**Velocity of Common Point Normal Velocity, Vcn, and Relative Normal, Vcrn**

$$V_{cn1}(\varepsilon, \beta) := V_{n1}(\varepsilon, \beta) + d_c \cdot \Omega_1(\varepsilon, \beta) \cdot \text{deg} \quad \text{Eq. 6.45} \quad V_{ct1}(\varepsilon, \beta) := V_{t1}(\varepsilon, \beta) - d_d \cdot \Omega_1(\varepsilon, \beta) \cdot \text{deg}$$

$$V_{cn2}(\varepsilon, \beta) := V_{n2}(\varepsilon, \beta) - d_a \cdot \Omega_2(\varepsilon, \beta) \cdot \text{deg} \quad V_{ct2}(\varepsilon, \beta) := V_{t2}(\varepsilon, \beta) + d_b \cdot \Omega_2(\varepsilon, \beta) \cdot \text{deg}$$

$$V_{crn}(\varepsilon, \beta) := V_{n1}(\varepsilon, \beta) + d_c \cdot \text{deg} \cdot \Omega_1(\varepsilon, \beta) - V_{n2}(\varepsilon, \beta) + d_a \cdot \text{deg} \cdot \Omega_2(\varepsilon, \beta) \quad \text{Consistency check for } \varepsilon$$

$$-V_{crn}(\varepsilon, \beta) \cdot v_{crn}^{-1} = 0.355$$

**Kinetic Energy Lost by Both and Individual Vehicles**

$$KE_{initial} := \frac{1}{2} \cdot [m_1 \cdot v_1^2 + m_2 \cdot v_2^2 + I_1 \cdot (\omega_1 \cdot \text{deg})^2 + I_2 \cdot (\omega_2 \cdot \text{deg})^2] = 1.074 \times 10^5 \cdot \text{ft} \cdot \text{lb}$$

### Definition of KE

$$KE_{final}(\epsilon, \beta) := \frac{1}{2} \cdot [m_1 \cdot V_1(\epsilon, \beta)^2 + m_2 \cdot V_2(\epsilon, \beta)^2 + I_1 \cdot (\Omega_1(\epsilon, \beta) \cdot \text{deg})^2 + I_2 \cdot (\Omega_2(\epsilon, \beta) \cdot \text{deg})^2]$$

$$\Delta KE_{loss}(\epsilon, \beta) := KE_{initial} - KE_{final}(\epsilon, \beta)$$

$$T_{lo}(\epsilon, \beta) := \frac{1}{2} \cdot m_C \cdot q(\epsilon, \beta) \cdot v_m^2 \cdot (1 + \epsilon)$$

Eq 6.56

### **Total Impact Loss, $T_L$**

$$T_L(\epsilon, \beta) := T_{lo}(\epsilon, \beta) \cdot \left[ 2 + 2 \cdot \mu(\epsilon, \beta) \cdot r - (1 + \epsilon) \cdot q(\epsilon, \beta) \cdot \left[ 1 + \mu(\epsilon, \beta)^2 + \left( \frac{m_C \cdot d_e(\epsilon, \beta)^2}{m_1 \cdot k_1^2} + \frac{m_C \cdot d_f(\epsilon, \beta)^2}{m_2 \cdot k_2^2} \right) \right] \right]$$

$$\%KE_{loss}(\epsilon, \beta) := \frac{100(\Delta KE_{loss}(\epsilon, \beta))}{KE_{initial}}$$

$$\text{AngularE}_f(\epsilon, \beta) := \frac{100}{2} \cdot \left[ I_1 \cdot \left( \Omega_1(\epsilon, \beta) \cdot \frac{\pi}{180} \right)^2 + I_2 \cdot \left( \Omega_2(\epsilon, \beta) \cdot \frac{\pi}{180} \right)^2 \right]$$

$$\%KE_{loss}(0, 1) = 31.457$$

$$\Delta KE_{loss}(0, 1) = 3.38 \times 10^4 \cdot \text{ft} \cdot \text{lb}$$

$$T_L(0, 1) = 3.38 \times 10^4 \cdot \text{ft} \cdot \text{lb}$$

$$E_{NormalCrush}(\epsilon, \beta) := P_{n1}(\epsilon, \beta) \cdot \frac{[(V_{cn2}(\epsilon, \beta) - V_{cn1}(\epsilon, \beta)) + v_{cn2} - v_{cn1}]}{2}$$

$$E_{NormalCrush}(0.5, 0.5) = 1.412 \times 10^4 \cdot \text{ft} \cdot \text{lb}$$

$$E_{TangentialCrush}(\epsilon, \beta) := P_{t1}(\epsilon, \beta) \cdot \frac{[(V_{ct2}(\epsilon, \beta) - V_{ct1}(\epsilon, \beta)) + v_{ct2} - v_{ct1}]}{2}$$

$$E_{NormalCrush}(0, 1) + E_{TangentialCrush}(0, 1) = 3.38 \times 10^4 \cdot \text{ft} \cdot \text{lb}$$

$$\epsilon_{PIM}(\epsilon, \beta) := \sqrt{\frac{E_{NormalCrush}(\epsilon, \beta)}{T_L(0, \beta)}}$$

$$PCETan(\epsilon, \beta) := 100 \cdot E_{TangentialCrush}(\epsilon, \beta) \cdot \%KE_{loss}(\epsilon, \beta)^{-1}$$

$$\Gamma_o := \Gamma \cdot \frac{\pi}{180}$$

### **Calculate Final X,Y Velocities and Momenta from $\Gamma$ & Final Normal and Tangential Velocities**

$$V_{x1}(\epsilon, \beta) := V_{n1}(\epsilon, \beta) \cdot \cos(\Gamma_o) - V_{t1}(\epsilon, \beta) \cdot \sin(\Gamma_o)$$

$$V_{y1}(\epsilon, \beta) := V_{n1}(\epsilon, \beta) \cdot \sin(\Gamma_o) + V_{t1}(\epsilon, \beta) \cdot \cos(\Gamma_o)$$

$$V_{x2}(\epsilon, \beta) := V_{n2}(\epsilon, \beta) \cdot \cos(\Gamma_o) - V_{t2}(\epsilon, \beta) \cdot \sin(\Gamma_o)$$

$$V_{y2}(\epsilon, \beta) := V_{n2}(\epsilon, \beta) \cdot \sin(\Gamma_o) + V_{t2}(\epsilon, \beta) \cdot \cos(\Gamma_o)$$

$$P_{x1}(\epsilon, \beta) := m_1 \cdot (V_{x1}(\epsilon, \beta) - v_{x1})$$

$$P_{y1}(\epsilon, \beta) := m_1 \cdot (V_{y1}(\epsilon, \beta) - v_{y1})$$

$$P_{x1}(\epsilon, \beta) = 1749.993 \cdot \text{s} \cdot \text{lb}$$

$$P_{y1}(\epsilon, \beta) = 896.544 \cdot \text{s} \cdot \text{lb}$$

$$V_{x1}(\epsilon, \beta) = -6.126 \frac{\text{ft}}{\text{s}}$$

$$V_{y1}(\epsilon, \beta) = 12.79 \frac{\text{ft}}{\text{s}}$$

$$V_{x2}(\epsilon, \beta) = -11.498 \frac{\text{ft}}{\text{s}}$$

$$V_{y2}(\epsilon, \beta) = 25.199 \frac{\text{ft}}{\text{s}}$$

### **Calculate PDOF**

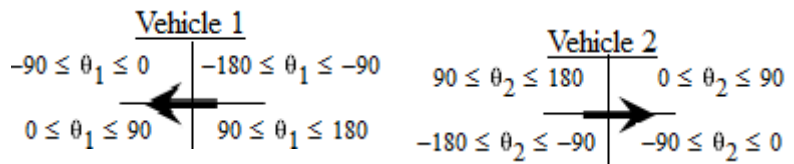
RICSAC 9 is a 90 degree collision. Angles are CCW. See drawing on pg. 2.

**Sign Conventions:** <https://www.brachengineering.com/content/vcrware-samples/planar-impact-mechanics.pdf>

The Principal Direction of Force, PDOF is the Line of Action of the Total Impulse

$$AT(\epsilon, \beta) := \frac{180}{\pi} \cdot \text{atan} \left( \left| \frac{P_{y1}(\epsilon, \beta)}{P_{x1}(\epsilon, \beta)} \right| \right)$$

$$AT(\epsilon, \beta) = 27.127$$



$$t1 := \text{if}(\theta_1 \geq 0, \theta_1, \theta_1 + 360)$$

$$t2 := \text{if}(\theta_2 \geq 0, \theta_2, \theta_2 + 360)$$

$$PDOF1(\epsilon, \beta) := \text{if}[(t1 - PF1(\epsilon, \beta)) > 180, t1 - PF1(\epsilon, \beta) - 360, IFP1(\epsilon, \beta)]$$

$$PDOF2(\epsilon, \beta) := \text{if}(180 + PF2(\epsilon, \beta) \leq -180, 180 + PF2(\epsilon, \beta) + 360, IFP2(\epsilon, \beta))$$

$$PDOF1(\epsilon, 1) = -27.127$$

$$PDOF2(\epsilon, 1) = 62.873$$

$$PDOF1(\epsilon, 1) - PDOF2(\epsilon, 1) = -90$$

### **Brach Planar Collision Model Assumptions:**

There is a common crush plane at a point, C.

The time duration of an Impact is small, thus changes in mass centers and angles are small.

Collision forces much greater than any of the other forces, e.g. friction, and very short.

Use Impulse and conservation of momentum --> no integration is necessary.

Preimpact dimensions are used. Changes in angular momentum equal to moments of impulses.

Newton's equations not enough information. Must use normal and tangential impact processes,  $\epsilon$  &  $\mu$

Find Six unknowns, Given 4 eqs. Two Constraints: Coefficients of Restitution,  $\epsilon$  and Impulse Ratio,

Therefore, to get post velocities, we must have knowledge of  $\epsilon$  and an estimate of  $\mu$

## **Find Optimal Solution: Least Error for V & $\Omega$**

Found by a Least Square Fit between the known Final RICSAC Velocities and Calculated Values

### **Known Final Velocities for RICSAC 9**

$$\begin{array}{lll} V_{xk1} := -5.1 \frac{\text{ft}}{\text{s}} & V_{yk1} := 17.6 \frac{\text{ft}}{\text{s}} & \Omega_{k1} := -180 \cdot \frac{1}{\text{s}} \\ V_{xk2} := -7.3 \frac{\text{ft}}{\text{s}} & V_{yk2} := 20.2 \frac{\text{ft}}{\text{s}} & \Omega_{k2} := 45 \cdot \frac{1}{\text{s}} \end{array}$$

### **Define Least Squares Error Functions, $V_{\text{error}}(\epsilon)$ and $\Omega_{\text{error}}(\beta)$**

$$V_{\text{error}}(\epsilon, \beta) := \left[ (V_{x1}(\epsilon, \beta) - V_{xk1})^2 + (V_{y1}(\epsilon, \beta) - V_{yk1})^2 + (V_{x2}(\epsilon, \beta) - V_{xk2})^2 + (V_{y2}(\epsilon, \beta) - V_{yk2})^2 \right]$$

$$\text{Guess} \quad \epsilon := 0.1 \quad \beta := 1$$

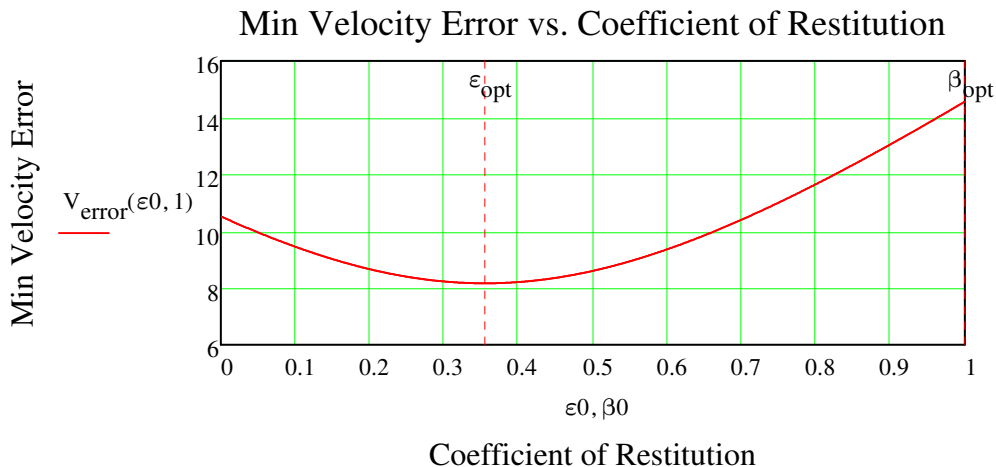
#### **Constraints**

$$\text{Given} \quad \epsilon \geq 0 \quad \beta \leq 1$$

### **Use Mathcad Minimize() Function to Find Values of $\epsilon_{\text{opt}}$ and $\beta_{\text{opt}}$ for Minimum V & $\Omega$ Error**

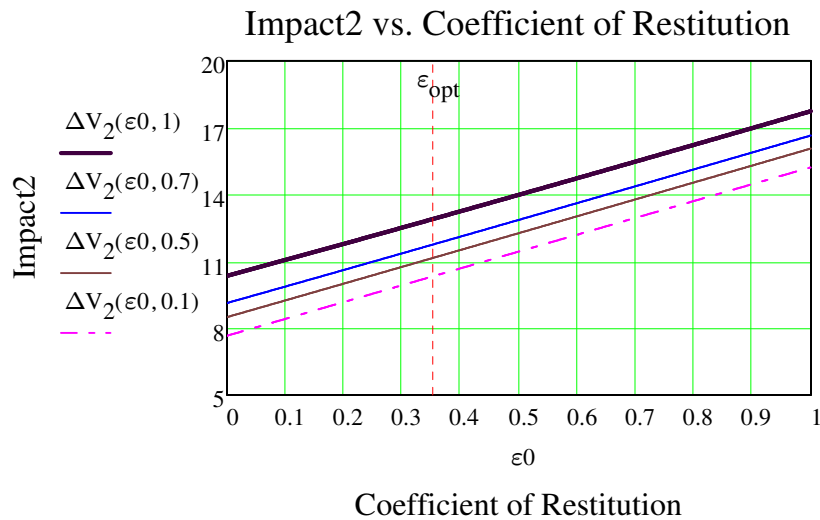
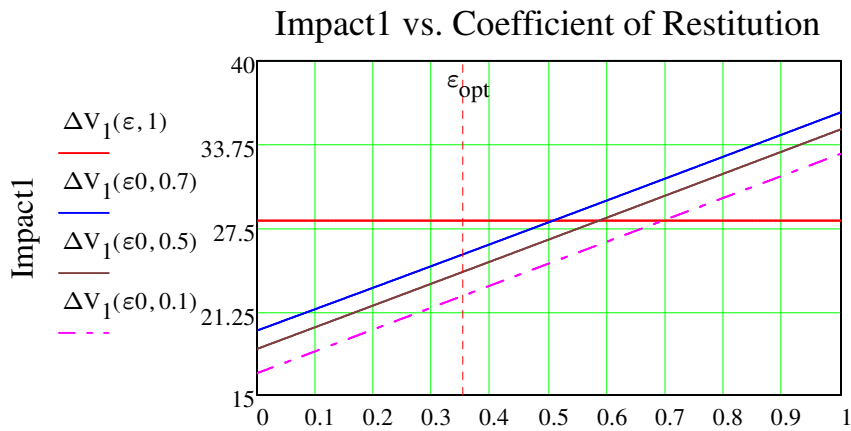
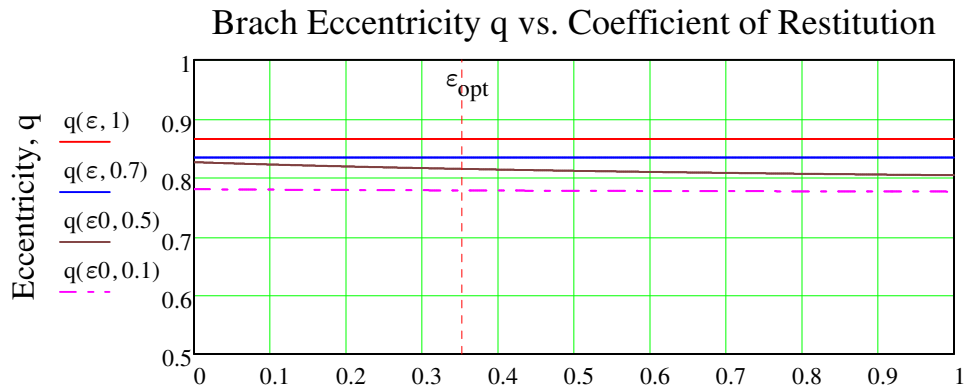
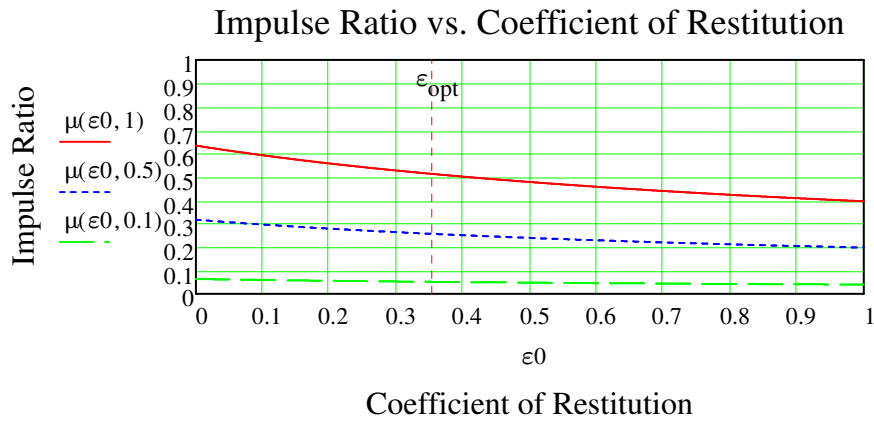
$$\epsilon \beta_{\text{opt2}} := \text{Minimize}(V_{\text{error}}, \epsilon, \beta) \quad \epsilon \beta_{\text{opt2}} = \begin{pmatrix} 0.355 \\ 1 \end{pmatrix} \quad \epsilon_{\text{opt}} := \epsilon \beta_{\text{opt2}_0} \quad \beta_{\text{opt}} := \epsilon \beta_{\text{opt2}_1}$$

### **Plot Least Squares Error to Find Optimum $\epsilon$ ( $\epsilon_{\text{opt}}$ ) and $\beta_{\text{opt}}$ for Minimum V and $\Omega$ Error**

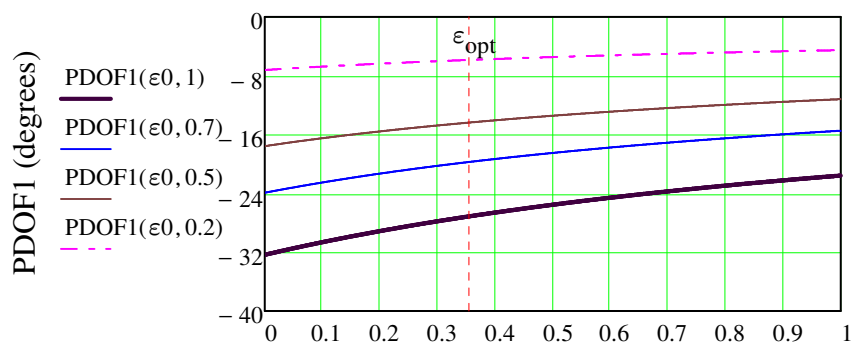


### **Plot Variation: Impulse Ratio, Eccentricity, Impulse, PDOF, Angular Velocity, & % KE Loss**

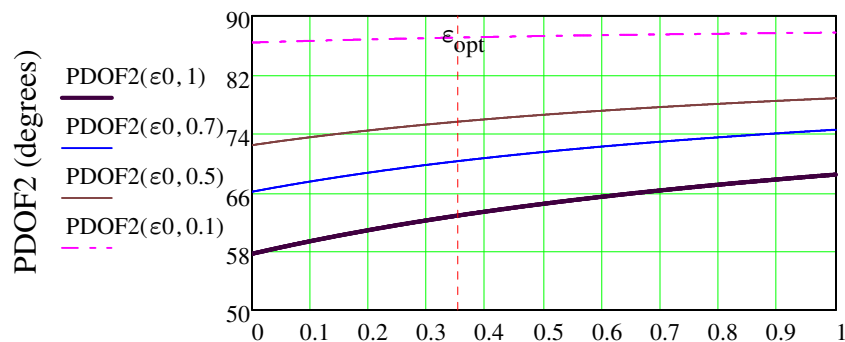
# Versus $\varepsilon$ for Various Values of $\beta$ (1, 0.7, 0.5, 0.1)



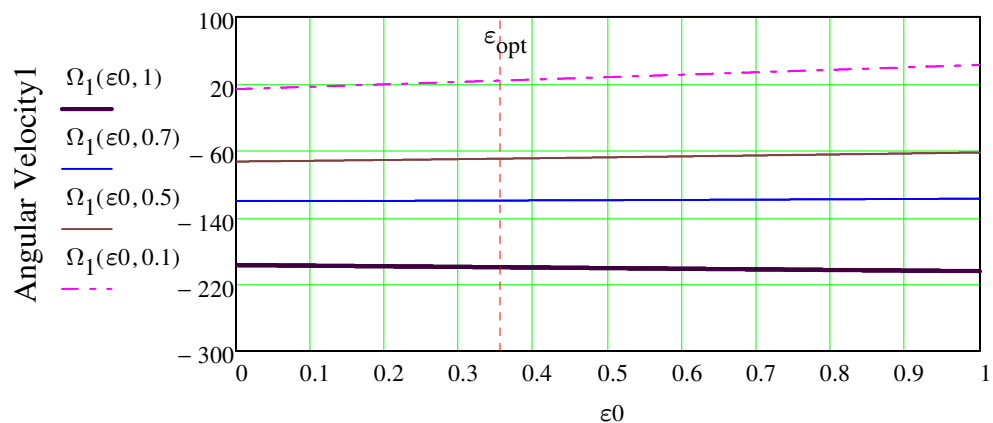
### PDOF1 vs. Coefficient of Restitution



### PDOF2 vs. Coefficient of Restitution

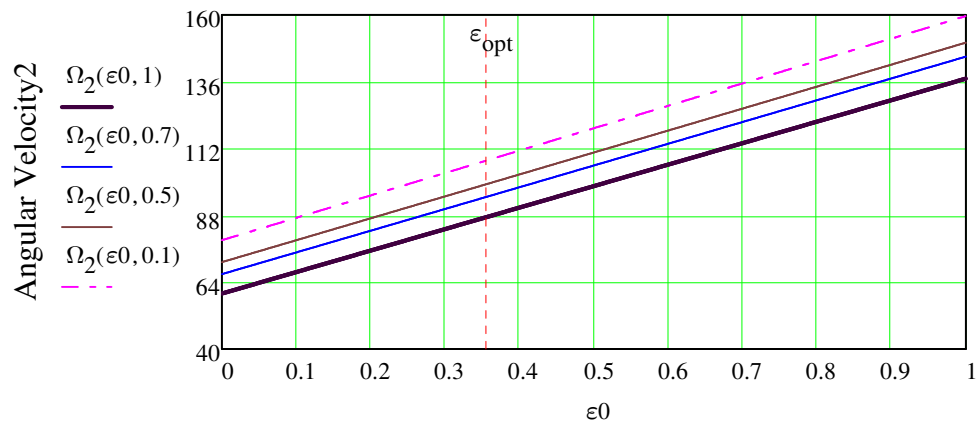


### Angular Velocity1 vs. Coefficient of Restitution



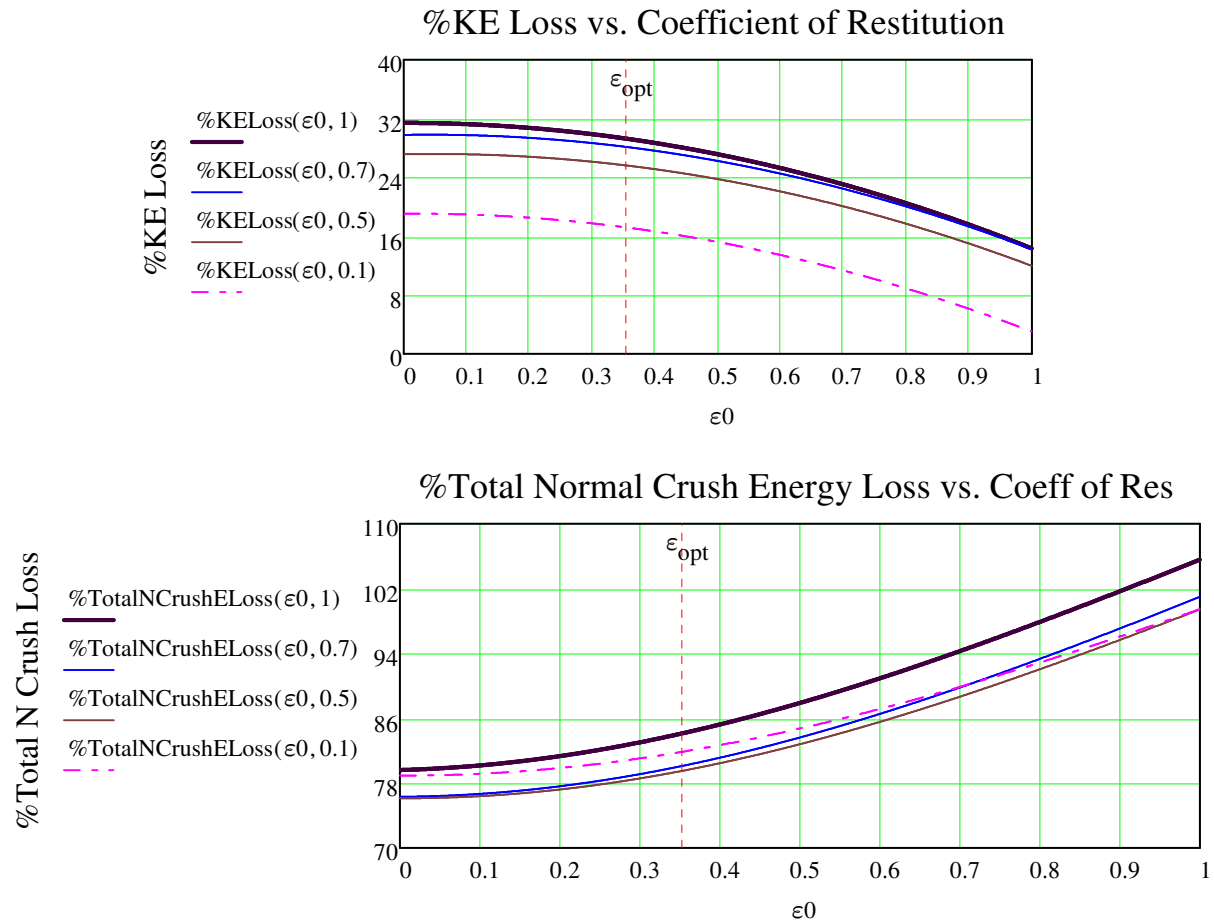
### Coefficient of Restitution

### Angular Velocity2 vs. Coefficient of Restitution





Note: In a Planar Collision a Coefficient of Restitution of 0 can still result in After Collision KE, because of Final Angular Momentum  $\Omega$  and Tangential Impulse.



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